Classifications of Simple Novikov Algebras and Their Irreducible Modules of Characteristic 0[1]

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Abstract

In this paper, we first present a classification theorem of simple infinite-dimensional Novikov algebras over an algebraically closed field with characteristic 0. Then we classify all the irreducible modules of a certain infinite-dimensional simple Novikov algebras with an idempotent element whose left action is locally finite.

1   Introduction

A *left-symmetric algebra* is an algebra whose associators are left symmetric. The commutator algebra associated with a left-symmetric algebra forms a Lie algebra. Left-symmetric algebras play a fundamental role in the theory of affine manifolds (cf. [A], [FD]). Kim [K1-2] classified all the left-symmetric algebras whose commutator Lie algebra are of small dimensions. Left-symmetric rings were studied by Kleinfeld [Kl]. A real finite-dimensional left-symmetric algebra $\mathcal{G}$ with $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$ is trivial (cf. [H]). Moreover, a left finite-dimensional left-symmetric algebra is trvial if its commutator Lie algebra is semi-simple over a field with characteristic 0 by Whitehead's Lemma. Over a field with characteristic p, Burde proved that there exist finite-dimensional left-symmetric algebras whose commutator Lie algebras are classical simple Lie algebras with dimension divisible by p or nonrestricted simple Lie algebras of Cartan type, and Shen [S] found some special left-symmetric algebras whose commutator Lie algebras are Witt algebras.

A *Novikov algebra* is a left-symmetric algebra whose right multiplication operators are mutually commutative. Novikov algebras appeared in the work of Gel'fand and Dorfman [GDo], corresponding to certain type of Hamiltonian operators. Balinskii and Novikov [BN] found the same algebraic structure in connection with Poisson brackets of hydrodynamic type. The abstract study of Novikov algebras was started by Zel'manov [Z] and Filipov [F]. The term "Novikov algebra" was given by Osborn [O1].

In this paper, we first present a classification theorem of simple infinite-dimensional Novikov algebras over an algebraically closed field with characteristic 0. Then we classify

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all the irreducible modules of a certain infinite-dimensional simple Novikov algebras with an idempotent element whose left action is locally finite.

To our best knowledge, there are no relatively complete classification results on simple left-symmetric algebras. However, we do have some relatively complete classification results on simple Novikov algebras. Zel'manov [Z] proved that a finite-dimensional simple Novikov algebra over an algebraically closed field with characteristic 0 is one-dimensional. Osborn [O1] proved that for any finite-dimensional simple Novikov algebra over a perfect field with characteristic $p > 2$, the associated commutator Lie algebra is isomorphic to a rank-one Witt algebra. These Witt algebras play fundamental roles in Lie algebras over a field with prime characteristic and are also important in other mathematical fields. An element e of a Novikov algebra $(\mathcal{N}, \circ)$ over a field $\mathbb{F}$ is called *idempotent* if $e \circ e \in \mathbb{F}e$. Moreover, Osborn [O2] classified finite-dimensional simple Novikov algebras with an idempotent element over an algebraically closed field with characteristic $p > 2$. We gave in [X1] a complete classification of finite-dimensional simple Novikov algebras over an algebraically closed field with characteristic $p > 2$ without any conditions. Although the classification problem of finite-dimensional irreducible modules of its associated rank-one Witt algebra is still open, finite-dimensional irreducible modules of a finite-dimensional simple Novikov algebra over an algebraically closed field with characteristic $p > 2$ were completely determined in [X1].

In [X1], we also introduced "Novikov-Poisson algebras," which are analogues of (Lie) Poisson algebras, and their tensor theory. Structures of Novikov-Poisson algebras have given us a better picture on simple Novikov algebras and their modules. Before our work [X2], all the known simple Novikov algebras (cf. [F], [Z], [O1-3], [X1]) have an idempotent element. A natural question was whether there exist simple Novikov algebras without idempotent elements. In [X2], a large family of simple Novikov algebras without idempotent elements were constructed through Novikov-Poisson algebras. Besides, a large class of Novikov-Poisson algebras are NX-bialgebras, which determine certain Hamiltonian superoperators of one supervariable (cf. [X3]).

Osborn [O3] gave a classification of infinite-dimensional simple Novikov algebras with an idempotent element, assuming the existence of generalized-eigenspace decomposition with respect to its left multiplication operator. There are four fundamental mistakes in his classification. The first is using of Proposition 2.6 (d) in [O1] with $\beta \neq 0$, which was misproved. The second is that the eigenspace A_0 in Lemma 2.12 of [O3] does not form a field when $b = 0$ with respect to the Novikov algebraic operation. The third is that A_0 may not be a perfect field when $b \neq 0$. The fourth is that the author forgot the case $b = 0$ and $\Delta = \{0\}$ in Lemma 2.8. In addition to these four mistakes, there are gaps in the arguments of classification in [O3]. It seems that one can not draw any conclusions of the classifications based on the arguments in [O3]. In [O5], Osborn have given certain properties of modules of infinite-dimensional simple Novikov algebras with an idempotent

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element.

A linear transformation T of a vector space V is called *locally finite* if the subspace

$$\sum_{m=0}^{\infty} \mathbb{F}T^m(v) \quad \text{is finite-dimensional for any} \quad v \in V. \tag{1.1}$$

An element u of a Novikov algebra $\mathcal{N}$ is called *left locally finite* if its left multiplication operator L_u is locally finite.

The aim of this paper is to classify infinite-dimensional simple Novikov algebras over an algebraically field $\mathbb{F}$ with characteristic 0, which contain a left locally finite element e whose right multiplication operator R_e is a constant map and left multiplication operator is surjective if $R_e = 0$ (see Theorem 3.4), and to classify all the irreducible modules of a certain infinite-dimensional simple Novikov algebras with an idempotent element whose left action is locally finite (see Theorem 4.3).

Throughout this paper, all the vector space are assumed over a field $\mathbb{F}$ with characteristic 0. Denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the additive semi-group of nonnegative intgers.

The paper is organized as follows. Section 2 is a preparation for our classification of algebras, where we discuss "homological group algebras" and their connection with simple Novikov algebras. The classification of simple Novikov algebras is given in Section 3. In Section 4, we classify the irreducible modules.

2 Homological Group Algebras

In this section, we shall introduce the notion "homological group algebra" and use it to construct a family of simple Novkov algebras larger than the one we obtained in Section 2 of [X2]

Definition 2.1. Let G be a group and let $\mathbb{F}_1$ be any field. Set

$$\mathbb{F}_1^{\times} = \mathbb{F}_1 \setminus \{0\}. \tag{2.1}$$

We view $\mathbb{F}_1^{\times}$ as the multiplication group of $\mathbb{F}_1$. A map $f : G \times G \to \mathbb{F}_1^{\times}$ is called a *two-cycle* if

$$f(g_1, g_2)f(g_1g_2, g_3) = f(g_1, g_2g_3)f(g_2, g_3) \quad \text{for} \quad g_1, g_2, g_3 \in G. \tag{2.2}$$

We denote the set of two-cycles by $c^2(G, \mathbb{F}_1^{\times})$. The multiplication on $c^2(G, \mathbb{F}_1^{\times})$ is defined by

$$(f_1 f_2)(g_1, g_2) = f_1(g_1, g_2)f_2(g_1, g_2) \quad \text{for} \quad f_1, f_2 \in c^2(G, \mathbb{F}_1^{\times}), \ g_1, g_2 \in G. \tag{2.3}$$

With respect to the above operation, $c^2(G, \mathbb{F}_1^{\times})$ forms an abelian group. For any map $\eta : G \to \mathbb{F}_1^{\times}$, we define

$$d_\eta(g_1, g_2) = \eta(g_1g_2)\eta(g_1)^{-1}\eta(g_2)^{-1} \quad g_1, g_2 \in G. \tag{2.4}$$

Then

$$
\begin{aligned}
d_\eta(g_1, g_2)d_\eta(g_1g_2, g_3) &= \eta(g_1g_2)\eta(g_1)^{-1}\eta(g_2)^{-1}\eta(g_1g_2g_3)\eta(g_1g_2)^{-1}\eta(g_3)^{-1} \\
&= \eta(g_1)^{-1}\eta(g_2)^{-1}\eta(g_3)^{-1}\eta(g_1g_2g_3) \\
&= \eta(g_1g_2g_3)\eta(g_1)^{-1}\eta(g_2g_3)^{-1}\eta(g_2g_3)\eta(g_2)^{-1}\eta(g_3)^{-1} \\
&= d_\eta(g_1, g_2g_3)d_\eta(g_2, g_3) \tag{2.5}
\end{aligned}
$$

for $g_1, g_2, g_3 \in G$. So $d_\eta \in c^2(G, \mathbb{F}_1^\times)$. We call d_η a *two-boundary*. The set $b^2(G, \mathbb{F}_1^\times)$ of two-boundaries forms a subgroup of $c^2(G, \mathbb{F}_1^\times)$. We define the *second cohomology group*:

$$
H^2(G, \mathbb{F}_1^\times) = c^2(G, \mathbb{F}_1^\times)/b^2(G, \mathbb{F}_1^\times). \tag{2.6}
$$

For any $f \in c^2(G, \mathbb{F}_1^\times)$, we define a *homological group algebra* $\mathbb{F}_1[G]_f$ to be a vector space with a basis $\{e_g \mid g \in G\}$ and its algebaric operation "$\cdot$" defined by

$$
e_{g_1} \cdot e_{g_2} = f(g_1, g_2)e_{g_1g_2} \qquad \text{for} \ \ g_1, g_2 \in \Gamma. \tag{2.7}
$$

Expression (2.2) implies the associativity of $(\mathbb{F}_1[G]_f, \cdot)$. For any map $\eta : G \to \mathbb{F}_1^\times$, $\{\eta(g)^{-1}e_g \mid g \in \Gamma\}$ is also a basis of $\mathbb{F}_1[G]_f$. Moreover,

$$
(\eta(g_1)^{-1}e_{g_1}) \cdot (\eta(g_2)^{-1}e_{g_2}) = f(g_1, g_2)\eta(g_1g_2)\eta(g_1)^{-1}\eta(g_2)^{-1}(\eta(g_1g_2)^{-1}e_{g_1g_2}) \tag{2.8}
$$

for $g_1, g_2 \in G$. This shows that f and fd_η define isomorphic homological group algebras. So each element in $H^2(G, \mathbb{F}_1)$ corresponds to a homological group algebra over $\mathbb{F}_1$. Another factor causing an isomorphisms between two homological group algebras is the automorphism group $\mathrm{Aut}\, G$ of G. For $\sigma \in \mathrm{Aut}\, G$, we define its action on $c^2(G, \mathbb{F}_1)$ by

$$
(\sigma f)(g_1, g_2) = f(\sigma^{-1}g_1, \sigma^{-1}g_2) \qquad \text{for} \ \ f \in c^2(G, \mathbb{F}_1), \ g_1, g_2 \in G. \tag{2.9}
$$

This provides a group action of $\mathrm{Aut}\, G$ on $c^2(G, \mathbb{F}_1)$. Similarly, we define a group action of $\mathrm{Aut}\, G$ on the set $\mathrm{Map}(G, \mathbb{F}_1^\times)$ of maps from G to $\mathbb{F}_1^\times$ by

$$
(\sigma\eta)(g) = \eta(\sigma^{-1}g) \qquad \text{for} \ \ \eta \in \mathrm{Map}(G, \mathbb{F}_1^\times), \ \sigma \in \mathrm{Aut}\, G, \ g \in G. \tag{2.10}
$$

Furthermore, for $\eta \in \mathrm{Map}(G, \mathbb{F}_1^\times)$ and $\sigma \in \mathrm{Aut}\, G$,

$$
\begin{aligned}
(\sigma d_\eta)(g_1, g_2) &= d_\eta(\sigma^{-1}g_1, \sigma^{-1}g_2) \\
&= \eta(\sigma^{-1}(g_1)\sigma^{-1}(g_2))\eta(\sigma^{-1}g_1)^{-1}\eta(\sigma^{-1}g_2)^{-1} \\
&= \eta(\sigma^{-1}(g_1g_2))\eta(\sigma^{-1}g_1)^{-1}\eta(\sigma^{-1}g_2)^{-1} \\
&= d_{\sigma\eta}(g_1, g_2) \tag{2.11}
\end{aligned}
$$

for $g_1, g_2 \in G$. Thus

$$
\sigma d_\eta = d_{\sigma\eta} \qquad \text{for} \ \ \eta \in \mathrm{Map}(G, \mathbb{F}_1^\times), \tag{2.12}
$$

which implies,

$$(\text{Aut } G)(b^2(G, \mathbb{F}_1)) = b^2(G, \mathbb{F}_1). \tag{2.13}$$

Thus we have an induced group action of $\text{Aut } G$ on $H^2(G, \mathbb{F}_1)$. We define the *absolute second cohomogy* of G over $\mathbb{F}_1$ as the set of $(\text{Aut } G)$-orbits:

$$H^{!,2}(G, \mathbb{F}_1) = H^2(G, \mathbb{F}_1)/(\text{Aut } G). \tag{2.14}$$

When G is torsion free, every invertible element in a homological group algebra $\mathbb{F}_1[G]_f$ is of form λe_g for some $\lambda \in \mathbb{F}_1^\times$ and $g \in G$. Hence we have:

Proposition 2.2. *There exists a one-to-one correspondence between the set of isomorphic classes of homological group algebras and absolute second cohomogy when the group is torsion-free.*

Note that there exists an identity element $\mathbf{1}$ in $c^2(G, \mathbb{F}_1)$ defined by

$$\mathbf{1}(g_1, g_2) = 1_{\mathbb{F}_1} \qquad \text{for } g_1, g_2 \in G. \tag{2.15}$$

In particular, $\mathbb{F}_1[G]_\mathbf{1}$ is the usual *group algebra* of G over $\mathbb{F}_1$. When the context is clear, we also use $\mathbf{1}$ to denote its image in $H^2(G, \mathbb{F}_1)$. Recall that a *perfect field* $\mathbb{F}_1$ is a field such that $x^n - \lambda = 0$ has a solution in $\mathbb{F}_1$ for any postive integer n and $\lambda \in \mathbb{F}_1$. A two cycle f of G is called *symmetric* if

$$f(g_1, g_2) = f(g_2, g_1) \qquad \text{for } g_1, g_2 \in G. \tag{2.16}$$

Note that when $g_2 = g_3 = 1$ in (2.2), we have

$$f(g_1, 1)f(g_1, 1) = f(g_1, 1)f(1, 1) \qquad \text{for } g_1 \in G, \tag{2.17}$$

eqivalently,

$$f(g_1, 1) = f(1, 1) \qquad \text{for } g_1 \in G. \tag{2.18}$$

Similarly, we get

$$f(1, 1) = f(1, g_3) \qquad \text{for } g_3 \in G \tag{2.19}$$

Moreover, we define $\zeta \in \text{Map}(G, \mathbb{F}^\times)$ by

$$\zeta(g) = f(1, 1)^{-\delta_{1,g}} \qquad \text{for } g \in G \tag{2.20}$$

Then

$$(fd_\zeta)(1, g) = (fd_\zeta)(g, 1) = 1 \qquad \text{for } g \in G \tag{2.21}$$

Proposition 5.4.4. *Suppose that G is a torsion-free abelian group and $f \in c^2(G, \mathbb{F}_1^\times)$ is symmetric. We have $f \in b^2(G, \mathbb{F}_1)$ if G is a free group or $\mathbb{F}_1$ is perfect.*

Proof. To prove the conclusion is equivalent to proving that $\mathbb{F}_1[G]_f$ is isomorphic to $\mathbb{F}_1[G]_\mathbf{1}$ for any symmetric $f \in c^2(G, \mathbb{F}_1^\times)$. For convenience, we use $+$ to denote the group

operation of the torsion-free abelian group G and 0 to denote 1_G. Let $f \in c^2(G, \mathbb{F}_1^\times)$ be a given symmetric element. By (2.17)-(2.21), we can assume

$$f(0, \alpha) = f(\alpha, 0) = 1 \qquad \text{for } \alpha \in G \tag{2.22}$$

Thus e_0 is an identity element of the algebra $\mathbb{F}_1[G]_f$, which is commutative by (2.16).

Assume that G is a free abelian group with a generator set Γ ($\mathbb{Z}$-basis). For any $\gamma \in \Gamma$, we let

$$\vartheta_\gamma = e_\gamma, \quad \vartheta_{-\gamma} = f(\gamma, -\gamma)^{-1} e_{-\gamma}. \tag{2.23}$$

Any element $\alpha \in G$ can be written as $\alpha = \sum_{\gamma \in \Gamma} \epsilon_\gamma n_\gamma \gamma$ with $n_\gamma \in \mathbb{N}$ and $\epsilon_\gamma \in \{1, -1\}$, we define

$$\vartheta_\alpha = \prod_{\gamma \in \Gamma} \vartheta_{\epsilon_\gamma \gamma}^{n_\gamma} \in \mathbb{F}_1 e_\alpha. \tag{2.24}$$

Then we have

$$\vartheta_\alpha \cdot \vartheta_\beta = \vartheta_{\alpha + \beta} \tag{2.25}$$

for $\alpha, \beta \in G$. So $\mathbb{F}_1[G]_f$ is isomorphic to $\mathbb{F}_1[G]_1$.

Suppose that $\mathbb{F}_1$ is perfect. We let $\vartheta_0 = e_0$. Assume that we have choosen

$$\{0 \neq \vartheta_\alpha \in \mathbb{F}_1 e_\alpha \mid \alpha \in G'\} \tag{2.26}$$

for a subgroup G' of G such that (2.25) holds for $\alpha, \beta \in G'$. Let $\gamma \in G \setminus G'$. If $\mathbb{Z}\gamma \bigcap G' = \{0\}$, we choose $\vartheta_{\pm\gamma}$ as in (2.23) and define

$$\vartheta_{\pm n\gamma + \alpha} = (\vartheta_{\pm\gamma})^n \cdot \vartheta_\alpha \qquad \text{for } n \in \mathbb{N}, \alpha \in G'. \tag{2.27}$$

Then (2.25) holds for $\alpha, \beta \in \mathbb{Z}\gamma + G'$. If $\mathbb{Z}\gamma \bigcap G' \neq \{0\}$, then

$$\mathbb{Z}\beta \bigcap G' = \mathbb{Z}m\gamma \tag{2.28}$$

for some positive integer m. Note that

$$e_\gamma^m = \lambda \vartheta_{m\gamma} \qquad \text{for some } 0 \neq \lambda \in \mathbb{F}_1. \tag{2.29}$$

Choose any $\mu \in \mathbb{F}_1$ such that $\mu^m = \lambda$ and define

$$\vartheta_\gamma = \mu e_\gamma, \quad \vartheta_{-\gamma} = \mu^{-1} f(\gamma, -\gamma)^{-1} e_{-\gamma}. \tag{2.30}$$

Then (2.27) is well defined. Again (2.25) holds for $\alpha, \beta \in \mathbb{Z}\gamma + G'$. Since (2.25) holds for $\alpha, \beta \in G'$ when $G' = \{0\}$, we can choose $\{0 \neq \vartheta_\alpha \in \mathbb{F}_1 e_\alpha \mid \alpha \in G\}$ such that (2.25) holds for $\alpha, \beta \in G$ by induction on subgroup G'. So $\mathbb{F}_1[G]_f$ is again isomorphic to $\mathbb{F}_1[G]_1$. $\qquad\square$

Let us now give a detailed definition of "Novikov algebra." A *Novikov algebra* is a vector space $\mathcal{N}$ with an algebraic operation $\circ$ such that

$$(u \circ v) \circ w = (u \circ w) \circ v, \tag{2.31}$$

$$(u \circ v) \circ w - u \circ (v \circ w) = (v \circ u) \circ w - v \circ (u \circ w) \tag{2.32}$$

for $u, v, w \in \mathcal{N}$. A subspace $\mathcal{I}$ of the Novikov algebra $\mathcal{N}$ is called an *left ideal* (*right ideal*) if $\mathcal{N} \circ \mathcal{I} \subset \mathcal{I}$ ($\mathcal{I} \circ \mathcal{N} \subset \mathcal{I}$). A subspace is called an *ideal* if it is a both left and right ideal. The algebra $\mathcal{N}$ is called *simple* if the only ideals of $\mathcal{N}$ are the *trivial ideals*: $\{0\}$, $\mathcal{N}$, and $\mathcal{N} \circ \mathcal{N} \neq \{0\}$.

Now we want to construct simple Novikov algebras. Let Δ be an additive subgroup of the base field of $\mathbb{F}$ and let $\mathbb{F}_1$ be an extension field of $\mathbb{F}$. For a symmetric element $f \in c^2(\Delta, \mathbb{F}_1^\times)$, we define $\mathcal{A}(\Delta, f, \mathbb{N})$ to be a vector space over $\mathbb{F}_1$ with a basis

$$\{u_{\alpha,j} \mid (\alpha, j) \in \Delta \times \mathbb{N}\} \tag{2.33}$$

and define an algebraic operation "$\cdot$" on $\mathcal{A}(\Delta, f, \mathbb{N})$ by

$$u_{\alpha_1,j_1} \cdot u_{\alpha_2,j_2} = f(\alpha_1, \alpha_2) u_{\alpha_1+\alpha_2,j_1+j_2} \qquad \text{for } (\alpha_1, j_1), (\alpha_2, j_2) \in \Delta \times \mathbb{N}. \tag{2.34}$$

Then $(\mathcal{A}(\Delta, f, \mathbb{N}), \cdot)$ forms a commutative associative algebra and

$$(\mathcal{A}(\Delta, f, \mathbb{N}), \cdot) \cong \mathbb{F}_1[\Delta]_f \otimes_{\mathbb{F}_1} \mathbb{F}_1[t] \tag{2.35}$$

as associative algebras. Moreover,

$$\mathcal{A}(\Delta, f, \{0\}) = \sum_{\alpha \in \Delta} \mathbb{F}_1 u_{\alpha,0} \tag{2.36}$$

forms a subalgebra that is isomorphic to $\mathbb{F}_1[\Delta]_f$.

Let $J \in \{\{0\}, \mathbb{N}\}$. The algebra $\mathcal{A}(\Delta, f, J)$ is defined as in the above. We define $\partial \in \mathrm{End}_{\mathbb{F}_1} \mathcal{A}(\Delta, f, J)$ by

$$\partial(u_{\alpha,j}) = \alpha u_{\alpha,j} + j u_{\alpha,j-1} \qquad \text{for } (\alpha, j) \in \Delta \times J. \tag{2.37}$$

We view $\mathcal{A}(\Delta, f, J)$ as an algebra over $\mathbb{F}$. Then ∂ is again a derivation. For any $\xi \in \mathcal{A}(\Delta, f, J)$, we define the algebraic operation $\circ_\xi$ on $\mathcal{A}(\Delta, f, J)$ over $\mathbb{F}$ by

$$u \circ_\xi v = u \cdot \partial(v) + \xi \cdot u \cdot v \qquad \text{for } u, v \in \mathcal{A}(\Delta, f, J). \tag{2.38}$$

By the proof of Theorem 2.9 in [X2], we have:

Proposition 2.3. *The algebra* $(\mathcal{A}(\Delta, f, J), \circ_\xi)$ *forms a simple Novikov algebra over the field* $\mathbb{F}$.

3 Classification of Algebras

In this section, we shall classify of infinite-dimensional simple Novikov algebras containing a left locally finite element e whose right multiplication operator R_e is a constant map and the left multiplication operator is surjective if $R_e = 0$.

Let $(\mathcal{N}, \circ)$ be a Novikov algebra. For $u \in \mathcal{N}$, we define the *left multiplication operator* L_u and the *right multiplication operator* R_u by

$$L_u(v) = u \circ v, \qquad R_u(v) = v \circ u \qquad \text{for } v \in \mathcal{N}. \tag{3.1}$$

Equation (2.31) implies

$$R_u R_v = R_v R_u, \qquad L_{u \circ v} = R_v L_u \qquad \text{for } u, v \in \mathcal{N}. \tag{3.2}$$

Define

$$[u, v]^- = u \circ v - v \circ u \qquad \text{for } u, v \in \mathcal{N}. \tag{3.3}$$

Then $(\mathcal{N}, [\cdot, \cdot]^-)$ form a Lie algebra, which is called the *commutator Lie algebra associated with the Novikov algebra* $(\mathcal{N}, \circ)$. Moreover, Equation (2.32) shows

$$L_{[u,v]^-} = [L_u, L_v], \qquad [L_u, R_v] = R_{u \circ v} - R_v R_u \qquad \text{for } u, v \in \mathcal{N} \tag{3.4}$$

For a fixed element $u \in \mathcal{N}$ and $\lambda \in \mathbb{F}$, we define

$$\mathcal{N}_{u,\lambda} = \{v \in \mathcal{N} \mid (R_u - \lambda)^m(v) = 0 \text{ for some } m \in \mathbb{N}\}. \tag{3.5}$$

Lemma 3.1 (Zel'manov, [Z]). *The subspace $\mathcal{N}_{u,\lambda}$ is an ideal of $\mathcal{N}$.*

Let M be a module of a Novikov algebra $(\mathcal{N}, \circ)$. We define the left action L_M and right action R_M by

$$L_M(u)(w) = u \circ w, \qquad R_M(u)(w) = w \circ u \qquad \text{for } u \in \mathcal{N}, \ w \in M. \tag{3.6}$$

Lemma 3.2 *Suppose that $\mathcal{N}$ has an element e such that*

$$e \circ e = \lambda e. \tag{3.7}$$

Then we have the following identity:

$$(R_M(e) - \lambda)^2 R_M(e) = 0. \tag{3.8}$$

Proof. Let $w \in M$. By (2.32),

$$(w \circ e) \circ e - w \circ (e \circ e) = (e \circ w) \circ e - e \circ (w \circ e), \tag{3.9}$$

which is equivalent to

$$(w \circ e) \circ e - \lambda w \circ e = \lambda e \circ w - e \circ (w \circ e) \tag{3.10}$$

by (2.31) and (3.7). Multiplying on the right by e, we obtain

$$((w \circ e) \circ e) \circ e - \lambda(w \circ e) \circ e = \lambda(e \circ w) \circ e - (e \circ (w \circ e)) \circ e, \tag{3.11}$$

which is equivalent to

$$((w \circ e) \circ e) \circ e - \lambda(w \circ e) \circ e = \lambda^2 e \circ w - \lambda e \circ (w \circ e) \qquad (3.12)$$

by (2.31). Subtracting $\lambda \times$ (3.10) from (3.12), we get

$$((w \circ e) \circ e) \circ e - 2\lambda(w \circ e) \circ e + \lambda^2 w \circ e = 0, \qquad (3.13)$$

equivalently,

$$(R_M(e) - \lambda)^2 R_M(e)(w) = 0. \qquad (3.14)$$

Since w is arbitrary, (3.8) follows from (3.14). □

Lemma 3.3. *Let $(\mathcal{N}, \circ)$ be a simple Novikov algebra with an element e such that $e \circ e = be$ with $b \in \mathbb{F}$ and L_e is locally finite. Set*

$$\mathcal{N}'_\alpha = \{u \in \mathcal{N} \mid (L_e - \alpha - b)^m(u) = 0 \text{ for some } m \in \mathbb{N}\} \qquad \text{for } \alpha \in \mathbb{F}, \qquad (3.15)$$

and denote

$$\Delta = \{\alpha \in \mathbb{F} \mid \mathcal{N}'_\alpha \neq \{0\}\}. \qquad (3.16)$$

Then

$$\mathcal{N} = \bigoplus_{\alpha \in \Delta} \mathcal{N}'_\alpha \qquad (3.17)$$

and

$$\mathcal{N}'_\alpha \circ \mathcal{N}'_\beta \subset \mathcal{N}'_{\alpha+\beta} \qquad \text{for } \alpha, \beta \in \Delta \text{ when } R_e|_{\mathcal{N}'_\alpha} = bId_{\mathcal{N}'_\alpha}, \qquad (3.18)$$

$$\mathcal{N}'_{-b} \circ \mathcal{N}'_\beta \subset \mathcal{N}'_{\beta-b} + \mathcal{N}'_{\beta-2b} \qquad \text{for } \beta \in \Delta. \qquad (3.19)$$

In particular, (3.18) implies

$$\mathcal{N}'_\alpha \circ \mathcal{N}'_\beta \subset \mathcal{N}'_{\alpha+\beta} \qquad \text{for } \alpha, \beta \in \Delta, \ \alpha \neq -b. \qquad (3.20)$$

Proof. Note that (3.17) follows from the local finiteness of L_e. We define

$$\partial = L_e - b. \qquad (3.21)$$

For $u, v \in \mathcal{N}$ such that $u \circ e = bu$, we get

$$\begin{aligned}
\partial(u \circ v) &= e \circ (u \circ v) - b(u \circ v) \\
&= (e \circ u) \circ v + u \circ (e \circ v) - (u \circ e) \circ v - b(u \circ v) \\
&= (e \circ u) \circ v + u \circ (e \circ v) - 2b(u \circ v) \\
&= (e \circ u - bu) \circ v + u \circ (e \circ v - bv) \\
&= \partial(u) \circ v + u \circ \partial(v) \qquad (3.22)
\end{aligned}$$

by (2.32). Observe that

$$\mathcal{N}'_\alpha = \{u \in \mathcal{N} \mid (\partial - \alpha)^m(n) = 0 \text{ for some } m \in \mathbb{N}\} \qquad \text{for } \alpha \in \Delta. \qquad (3.23)$$

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Suppose that $u, v \in \mathcal{N}$ satisfy $u \circ e = bu$ and

$$(\partial - \alpha)^{m_1}(u) = 0, \quad (\partial - \beta)^{m_2}(v) = 0 \qquad \text{for some } \alpha, \beta \in \Delta, \ m_1, m_2 \in \mathbb{N}. \tag{3.24}$$

$$
\begin{aligned}
&(\partial - \alpha - \beta)^{m_1 + m_2}(u \circ v) \\
={}& (\partial - \alpha - \beta)^{m_1 + m_2 - 1}(\partial(u \circ v) - (\alpha + \beta)u \circ v) \\
={}& (\partial - \alpha - \beta)^{m_1 + m_2 - 1}(\partial(u) \circ v + u \circ \partial(v) - (\alpha + \beta)u \circ v) \\
={}& (\partial - \alpha - \beta)^{m_1 + m_2 - 1}((\partial - \alpha)(u) \circ v + u \circ (\partial - \beta)(v)) \\
={}& \sum_{j=0}^{m_1 + m_2} \binom{m_1 + m_2}{j}(\partial - \alpha)^j(u)(\partial - \beta)^{m_1 + m_2 - j}(v) = 0.
\end{aligned}
\tag{3.25}
$$

Thus (3.18) holds. Furthermore, by linear algebra,

$$e \circ \mathcal{N}'_\alpha = \mathcal{N}'_\alpha \qquad \text{for } -b \neq \alpha \in \Delta. \tag{3.26}$$

By (2.31),

$$R_e|_{\mathcal{N}'_\alpha} = b\,\mathrm{Id}_{\mathcal{N}'_\alpha} \qquad \text{for } -b \neq \alpha \in \Delta. \tag{3.27}$$

Hence (3.20) is implied by (3.18).

For any $\alpha \in \mathbb{F}$, we set

$$\mathcal{N}_\alpha = \{u \in \mathcal{N} \mid L_e(u) = (\alpha + b)u\}. \tag{3.28}$$

Since $(R_e - b)(e) = 0$, we have

$$(R_e - b)^2 = 0 \tag{3.29}$$

by Lemmas 3.1 and 3.2. For any $u \in \mathcal{N}$, we have

$$
\begin{aligned}
e \circ (u \circ e - bu) &= (e \circ u) \circ e + u \circ (e \circ e) - (u \circ e) \circ e - be \circ u \\
&= (e \circ e) \circ u + bu \circ e - (u \circ e) \circ e - be \circ u \\
&= (bR_e - R_e^2)(u) \\
&= (b^2 - bR_e)(u) \\
&= -b(u \circ e - bu)
\end{aligned}
\tag{3.30}
$$

by (2.31), (2.32) and (3.29), and

$$(u \circ e - bu) \circ e = (R_e^2 - bR_e)(u) = (bR_e - b^2)(u) = b(u \circ e - bu) \tag{3.31}$$

by (3.29). So

$$(R_e - b)(\mathcal{N}) \subset \mathcal{N}_{-2b}, \quad R_e|_{(R_e - b)(\mathcal{N})} = b. \tag{3.32}$$

Let $u, v \in \mathcal{N}$ such that

$$(\partial + b)^{m_1}(u) = 0, \quad (\partial - \beta)^{m_2}(v) = 0 \qquad \text{for some } \beta \in \Delta, \ m_1, m_2 \in \mathbb{N}. \tag{3.33}$$

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By (3.25) and (3.32),

$$(\partial - \beta + 2b)^{m_2}[(\partial + b)^{j_1}(b - R_e)(u)] \circ (\partial - \beta)^{j_2}(v) = 0 \qquad \text{for } j_1, j_2 \in \mathbb{N}. \qquad (3.34)$$

Since

$$
\begin{aligned}
& (\partial - \beta + 2b)^{m_2}(\partial - \beta + b)(u \circ v) \\
= {} & (\partial - \beta + 2b)^{m_2}[e \circ (u \circ v) - \beta(u \circ v)] \\
= {} & (\partial - \beta + 2b)^{m_2}[(e \circ u) \circ v + u \circ (e \circ v) - (u \circ e) \circ v - \beta(u \circ v)] \\
= {} & (\partial - \beta + 2b)^{m_2}[(\partial + b)(u) \circ v + u \circ (e - \beta - b) \circ v) + (b - R_e)(u) \circ v] \\
= {} & (\partial - \beta + 2b)^{m_2}[(\partial + b)(u) \circ v + u \circ (\partial - \beta) \circ v)] \qquad (3.35)
\end{aligned}
$$

by (2.32) and (3.34), we have

$$
\begin{aligned}
& (\partial - \beta + 2b)^{m_2}(\partial - \beta + b)^{m_1 + m_2}(u \circ v) \\
= {} & \sum_{j=0}^{m_1 + m_2} \binom{m_1 + m_2}{j}(\partial - \beta + 2b)^{m_2}(\partial + b)^j(u)(\partial - \beta)^{m_1 + m_2 - j}(v) = 0. \qquad (3.36)
\end{aligned}
$$

Thus (3.19) holds. $\quad\square$

Note the arguments in the above proof also show the following properties on the eigenspaces:

$$\mathcal{N}_\alpha \circ \mathcal{N}_\beta \subset \mathcal{N}_{\alpha + \beta} \qquad \text{for } \alpha, \beta \in \Delta \text{ when } R_e|_{\mathcal{N}_\alpha} = b\mathrm{Id}_{\mathcal{N}_\alpha}, \qquad (3.37)$$

$$\mathcal{N}_{-b} \circ \mathcal{N}_\beta \subset \mathcal{N}_{\beta - b} + \mathcal{N}_{\beta - 2b} \qquad \text{for } \beta \in \Delta. \qquad (3.38)$$

In particular, (3.37) implies

$$\mathcal{N}_\alpha \circ \mathcal{N}_\beta \subset \mathcal{N}_{\alpha + \beta} \qquad \text{for } \alpha, \beta \in \Delta, \; \alpha \neq -b. \qquad (3.39)$$

Below, we shall re-establish a classification theorem, partially based on Osborn's arguments in [O3]. We assume that $\mathbb{F}$ is algebraically closed. The following is the first main theorem in this paper.

Theorem 3.4. *Suppose that $(\mathcal{N}, \circ)$ is an infinite-dimensional simple Novikov algebra with a left locally finite element e whose right multiplication operator R_e is a constant map and left multiplication operator is surjective if $R_e = 0$. Then there exist an additive subgroup Δ of $\mathbb{F}$, an extension field $\mathbb{F}_1$ of $\mathbb{F}$, a symmetric element $f \in c^2(\Delta, \mathbb{F}_1)$, $J \in \{\{0\}, \mathbb{N}\}$ and $\xi \in \mathbb{F}$ such that the algebra $(\mathcal{N}, \circ)$ is isomorphic to $(\mathcal{A}(\Delta, f, J), \circ_\xi)$ (cf. (2.38)).*

Proof. Let $(\mathcal{N}, \circ)$ be the Novikov algebra in the theorem. Assume

$$R_e = b\mathrm{Id}_{\mathcal{N}} \quad \text{with } b \in \mathbb{F}. \qquad (3.40)$$

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In particular,

$$e \circ e = be. \tag{3.41}$$

We shall use the notations and conclusions in the above lemma.

Let

$$\hat{\mathcal{N}} = e \circ \mathcal{N}. \tag{3.42}$$

Then

$$\hat{\mathcal{N}} \supset \sum_{-b \neq \alpha \in \Delta} \mathcal{N}'_\alpha \tag{3.43}$$

by (3.26). In fact, one can derive

$$R_e|_{\hat{\mathcal{N}}} = b\mathrm{Id}_{\hat{\mathcal{N}}} \tag{3.44}$$

from (3.41) by (3.31) without assumption (3.40). The assumption (3.40) is a replacement of (3.41) that was used in [O3] in the following Case 3 of our classification.

For

$$u \in \sum_{n=0}^{\infty} (R_\mathcal{N})^n (e) \ \text{ and } \ v \in \mathcal{N}, \tag{3.45}$$

Define

$$u \cdot (e \circ v) = u \circ v. \tag{3.46}$$

Then "$\cdot$": $\hat{\mathcal{N}} \times \hat{\mathcal{N}} \to \mathcal{N}$ is a commutative bilinear map by (2.31). For any $u, v, w \in \mathcal{N}$ such that $(e \circ v) \circ w \in \hat{\mathcal{N}}$, we have

$$
\begin{aligned}
(e \circ u) \cdot [(e \circ v) \cdot (e \circ w)] &= [(e \circ v) \cdot (e \circ w)] \cdot (e \circ u) \\
&= ((e \circ v) \circ w) \circ u \\
&= ((e \circ u) \circ v] \circ w \\
&= [(e \circ u) \cdot (e \circ v)] \cdot (e \circ w)
\end{aligned}
\tag{3.47}
$$

by (2.31) and the commutativity. So the map "$\cdot$" is associative. Furthermore, when $e \in \hat{\mathcal{N}}$,

$$e \cdot (e \circ u) = e \circ u \qquad \text{for } u \in \mathcal{N}. \tag{3.48}$$

Hence e is an identity element of $(\hat{\mathcal{N}}, \cdot)$ when $b \neq 0$.

We use the notation in (3.21). For $u, v \in \mathcal{N}$, by (3.22), we get

$$
\begin{aligned}
\partial[(e \circ u) \cdot (e \circ v)] &= \partial[(e \circ u) \circ v] \\
&= \partial(e \circ u) \circ v + (e \circ u) \circ \partial(v) \\
&= \partial(e \circ u) \circ v + (e \circ u) \cdot (e \circ \partial(v)) \\
&= \partial(e \circ u) \cdot (e \circ v) + (e \circ u) \cdot \partial(e \circ v).
\end{aligned}
\tag{3.49}
$$

So ∂ is a derivation with respect to $(\hat{\mathcal{N}}, \cdot)$. Moreover,

$$(e \circ u) \circ v = (e \circ u) \cdot (e \circ v) = (e \circ u) \cdot (\partial(v) + bv) \qquad \text{for } u, v \in \mathcal{N}. \tag{3.50}$$

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Furthermore, by (3.18) and the fact that $e \circ \mathcal{N}'_\alpha \subset \mathcal{N}'_\alpha$ for any $\alpha \in \Delta$ (cf. (3.16)), we have

$$\mathcal{N}'_\alpha \cdot \mathcal{N}'_\beta \subset \mathcal{N}'_{\alpha+\beta} \qquad \text{for } \mathcal{N}'_\alpha, \mathcal{N}'_\beta \subset \hat{\mathcal{N}}. \tag{3.51}$$

Case 1. $\mathcal{N} = \hat{\mathcal{N}}$.

Under this assumption, $(\mathcal{N}, \cdot)$ is ∂-simple commutative and associative algebra by (3.50) and the simplicity of $(\mathcal{N}, \circ)$, that is, the only ∂-invariant ideal of $(\mathcal{N}, \cdot)$ are $\mathcal{N}$ and $\{0\}$. Hence

$$(\mathcal{N}, \cdot) \cong \mathcal{A}(\Delta, f, \mathbb{N}) \tag{3.52}$$

(cf. (2.33), (2.34)) by Theorem 2.1 in [SXZ]. Therefore by (3.50),

$$(\mathcal{N}, \circ) \cong (\mathcal{A}(\Delta, f, \mathbb{N}), \circ_b). \tag{3.53}$$

Case 2. $\hat{\mathcal{N}} \neq \mathcal{N}$, $b \neq 0$ and

$$\mathcal{N}'_{-b} \subset \sum_{0, -b \neq \alpha \in \Delta} \mathcal{N}'_\alpha \circ \mathcal{N}'_{-\alpha-b} \tag{3.54}$$

(cf. (3.18)).

Replacing e by $b^{-1}e$, we can assume $b = 1$. If $-1 \notin \Delta$, then $\hat{\mathcal{N}} = \mathcal{N}$, which is in Case 1. So we asssume $-1 \in \Delta$. In this case, we can derive (3.40) from (3.41) and (3.54) by (2.31). For $0 \neq u \in \mathcal{N}_\alpha$ with $\alpha \in \Delta$, we set

$$I_u = \sum_{n=0}^{\infty} (R_\mathcal{N})^n(u). \tag{3.55}$$

Note that $e \circ u = (\alpha + 1)u \in I_u$. Suppose that

$$e \circ ((\cdots((u \circ v_1) \circ v_2) \cdots) \circ v_k) \in I_u \qquad \text{for } v_s \in \mathcal{N}, \ s = 1, ..., k. \tag{3.56}$$

Denote

$$w = (\cdots((u \circ v_1) \circ v_2) \cdots) \circ v_k. \tag{3.57}$$

For any $v_{k+1} \in \mathcal{N}$, we have

$$e \circ (w \circ v_{k+1}) = (e \circ w) \circ v_{k+1} + w \circ (e \circ v_{k+1}) - (w \circ e) \circ v_k \in I_u. \tag{3.58}$$

By induction on k, we have

$$e \circ I_u \subset I_u. \tag{3.59}$$

Hence

$$\hat{\mathcal{N}} \circ I_u = (e \circ \mathcal{N}) \circ I_u = (e \circ I_u) \circ \mathcal{N} \subset I_u. \tag{3.60}$$

Moreover, by (2.31), (3.43) and (3.54), I_u is an ideal of $\mathcal{N}$. Since $u = u \circ e \in I_u$, I_u is a nonzero ideal. The simplicity of $\mathcal{N}$ implies

$$I_u = \mathcal{N}. \tag{3.61}$$

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Note that
$$\hat{\mathcal{N}} \circ \mathcal{N}_{-1} = (e \circ \mathcal{N}) \circ \mathcal{N}_{-1} = (e \circ \mathcal{N}_{-1}) \circ \mathcal{N} = \{0\} \tag{3.62}$$
by (2.31) and (3.28). Moreover, (3.54) and (3.62) imply
$$\mathcal{N} \circ \mathcal{N}_{-1} = \{0\} \tag{3.63}$$
by (2.31). Suppose that
$$u \circ v = 0 \qquad \text{for some } 0 \neq u \in \mathcal{N}'_\alpha, \ 0 \neq v \in \mathcal{N}'_\beta. \tag{3.64}$$
Let m_1, m_2 be the minimal non-negative integers such that
$$(\partial - \alpha)^{m_1}(u) \neq 0, \ \ (\partial - \beta)^{m_2}(v) \neq 0, \ \ (\partial - \alpha)^{m_1+1}(u) = (\partial - \beta)^{m_2+1}(v) = 0 \tag{3.65}$$
(cf. (3.23)). By (3.25), we have
$$0 = (\partial - \alpha - \beta)^{m_1+m_2}(u \circ v) = \binom{m_1+m_2}{m_1}(\partial - \alpha)^{m_1}(u) \circ (\partial - \beta)^{m_2}(v). \tag{3.66}$$
So we have
$$(\partial - \alpha)^{m_1}(u) \circ (\partial - \beta)^{m_2}(v) = 0. \tag{3.67}$$
Observe that $(\partial - \alpha)^{m_1}(u) \in \mathcal{N}_\alpha$ by (3.65). Hence
$$\mathcal{N} \circ (\partial - \beta)^{m_2}(v) = I_{(\partial-\alpha)^{m_1}(u)} \circ (\partial - \beta)^{m_2}(v) = \{0\} \tag{3.68}$$
by (2.31), (3.55) and (3.61). Furthermore by (3.25) and (3.68), we have
$$0 = (\partial - \alpha - \beta)^{m_1+m_2-1}(u \circ v) = \binom{m_1+m_2-1}{m_1}(\partial - \alpha)^{m_1}(u) \circ (\partial - \beta)^{m_2-1}(v) \tag{3.69}$$
when $m_2 > 0$. By the above arguments, we get
$$\mathcal{N} \circ (\partial - \beta)^{m_2-1}(v) = \{0\} \tag{3.70}$$
when $m_2 > 0$. Continuing this process, we can prove
$$\mathcal{N} \circ v = 0. \tag{3.71}$$
In particular,
$$e \circ v = 0. \tag{3.72}$$
Hence
$$v \in \mathcal{N}_{-1}. \tag{3.73}$$
Therefore,
$$R_v \text{ is injective} \qquad \text{for } v \in \left(\bigcup_{\alpha \in \Delta} \mathcal{N}'_\alpha\right) \setminus \mathcal{N}_{-1}. \tag{3.74}$$

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For any $\alpha \in \Delta$, we pick $0 \neq u \in \mathcal{N}_\alpha$. Expressions (3.55) and (3.61) imply that there exists $\{v_s \in \mathcal{N}'_{\beta_s} \mid s = 1, ..., k\}$ such that

$$0 \neq (\cdots ((u \circ v_1) \circ v_2) \cdots) \circ v_k \in \mathcal{N}'_0. \tag{3.75}$$

Moreover, $v_s \notin \mathcal{N}_{-1}$ by (3.63) for each $s \in \{1, ..., k\}$ and $\sum_{j=1}^k \beta_j = -\alpha$. By (3.64),

$$0 \neq (\cdots (v_1 \circ v_2) \cdots) \circ v_k \in \mathcal{N}'_{-\alpha}. \tag{3.76}$$

So

$$-\alpha \in \Delta \qquad \text{for } \alpha \in \Delta. \tag{3.77}$$

Therefore, Δ forms an additive subgroup of $\mathbb{F}$ by (3.18) and the fact that $e \in \mathcal{N}_0$.

By (3.21) and (3.28),

$$e \circ u = u, \quad \partial(u) = 0 \qquad \text{for } u \in \mathcal{N}_0 \tag{3.78}$$

(recall $b = 1$). Moreover,

$$u \circ v = u \cdot (e \circ v) = u \cdot v \qquad \text{for } u \in \hat{\mathcal{N}}, \ v \in \mathcal{N}_0. \tag{3.79}$$

Let $u_1 \in \mathcal{N}'_\alpha$, $u_2 \in \mathcal{N}'_\beta$ with $-1 \neq \alpha, \beta \in \Delta$ and $v_1, v_2 \in \mathcal{N}_0$. When $\alpha + \beta \neq -1$, we have

$$
\begin{aligned}
(u_1 \circ v_1) \circ (u_2 \circ v_2) &= (u_1 \circ (u_2 \circ v_2)) \circ v_1 \\
&= (u_1 \cdot (\partial + 1)(u_2 \cdot v_2)) \cdot v_1 \\
&= (u_1 \cdot ((\partial + 1)(u_2) \cdot v_2)) \cdot v_1 \\
&= (u_1 \cdot (\partial + 1)(u_2)) \cdot v_2) \cdot v_1 \\
&= (u_1 \circ u_2) \cdot (v_2 \cdot v_1) \\
&= (u_1 \circ u_2) \cdot (v_1 \cdot v_2) \\
&= (u_1 \circ u_2) \circ (v_1 \circ v_2)
\end{aligned} \tag{3.80}
$$

by (2.31), (3.18), (3.47), (3.49), (3.50), (3.78) and (3.79). When $\alpha + \beta = -1$, we write $u_1 = e \circ w$ with $w \in \mathcal{N}'_\alpha$ and obtain

$$
\begin{aligned}
(u_1 \circ v_1) \circ (u_2 \circ v_2) &= ((e \circ w) \circ v_1) \circ (u_2 \circ v_2) \\
&= ((e \circ v_1) \circ (u_2 \circ v_2)) \circ w \\
&= [(e \circ u_2) \circ (v_1 \circ v_2)] \circ w \\
&= ((e \circ w) \circ u_2) \circ (v_1 \circ v_2) \\
&= (u_1 \circ u_2) \circ (v_1 \circ v_2)
\end{aligned} \tag{3.81}
$$

by (2.31) and (3.80). Now for $u_1 \in \mathcal{N}'_\alpha$, $u_2 \in \mathcal{N}'_{-\alpha-1}$, $u_3 \in \mathcal{N}'_\beta$ with $-1 \neq \alpha, \beta \in \Delta$ and $v_1, v_2 \in \mathcal{N}_0$, we have

$$
\begin{aligned}
[(u_1 \circ u_2) \circ v_1] \circ (u_3 \circ v_2) &= [(u_1 \circ v_1) \circ (u_3 \circ v_2)] \circ u_2 \\
&= [(u_1 \circ u_3) \circ (v_1 \circ v_2)] \circ u_2 \\
&= [(u_1 \circ u_2) \circ u_3] \circ (v_1 \circ v_2)
\end{aligned} \tag{3.82}
$$

by (2.31), (3.80) and (3.81). Hence we get

$$(u_1 \circ v_1) \circ (u_2 \circ v_2) = (u_1 \circ u_2) \circ (v_1 \circ v_2) \tag{3.83}$$

for $u_1 \in \mathcal{N}'_\alpha$, $u_2 \in \mathcal{N}'_\beta$ with $\alpha, \beta \in \Delta$, $\beta \neq -1$ and $v_1, v_2 \in \mathcal{N}_0$ by (3.54) and (3.80)-(3.82).

Observe

$$e \circ (u \circ v) = (e \circ u) \circ v + u \circ (e \circ v) - (u \circ e) \circ v = (e \circ u) \circ v \tag{3.84}$$

for $u \in \mathcal{N}$ and $v \in \mathcal{N}_0$ by (2.32), (3.40) and (3.78) (recall $b = 1$). Thus for $u_1 \in \mathcal{N}'_\alpha$ with $\alpha \neq -1$, $u_2 \in \mathcal{N}_{-1}$ and $v_1, v_2 \in \mathcal{N}_0$, we write $u_1 = e \circ w$ with $w \in \mathcal{N}'_\alpha$ and get

$$
\begin{aligned}
(u_1 \circ v_1) \circ (u_2 \circ v_2) &= ((e \circ w) \circ v_1) \circ (u_2 \circ v_2) \\
&= ((e \circ (u_2 \circ v_2)) \circ w) \circ v_1 \\
&= ((e \circ u_2) \circ v_2) \circ w) \circ v_1 \\
&= (((e \circ u_2) \circ w) \circ v_2) \circ (e \circ v_1) \\
&= (((e \circ w) \circ u_2) \circ e) \circ (v_2 \circ v_1) \\
&= (u_1 \circ u_2) \circ (v_1 \circ v_2) \tag{3.85}
\end{aligned}
$$

by (2.31), (3.83) and (3.84). Moreover, for $u_1 \in \mathcal{N}'_\alpha$, $u_2 \in \mathcal{N}'_{-\alpha-1}$ with $-1 \neq \alpha \in \Delta$, $u_3 \in \mathcal{N}'_{-1}$ and $v_1, v_2 \in \mathcal{N}_0$, we have (3.82). Therefore, by (3.54), (3.82) with $u_3 \in \mathcal{N}'_{-1}$, (3.83) and (3.85), we get

$$(u_1 \circ v_1) \circ (u_2 \circ v_2) = (u_1 \circ u_2) \circ (v_1 \circ v_2) \qquad \text{for} \quad u_1, u_2 \in \mathcal{N}, \ v_1, v_2 \in \mathcal{N}_0. \tag{3.86}$$

Let $0 \neq v \in \mathcal{N}_0$. Then $\mathcal{N} \circ v$ is a nonzero ideal of $\mathcal{N}$ by (3.86) with $v_1 = e$. Hence

$$\mathcal{N} \circ v = \mathcal{N}, \tag{3.87}$$

which implies

$$\mathcal{N}'_0 \circ v = \mathcal{N}'_0 \tag{3.88}$$

by (3.54). Thus there exists $u \in \mathcal{N}'_0$ such that

$$u \circ v = e. \tag{3.89}$$

By (3.78),
$$0 = \partial(u \circ v) = \partial(u) \circ v + u \circ \partial(v) = \partial(u) \circ v. \tag{3.90}$$

So $\partial(u) = 0$ by (3.74). Thus $u \in \mathcal{N}_0$ by (3.21) and (3.28). Therefore $(\mathcal{N}_0, \circ)$ forms a field by (3.79) and the commutativity and associativity of $(\mathcal{N}_0, \cdot)$. Moreover, $\mathcal{N}'_\alpha$ and $\mathcal{N}_\alpha$ are right vector space over $\mathcal{N}_0$ for $\alpha \in \Delta$ by (3.86) with $u_2 = e$. For $1 \neq \alpha \in \Delta$ and $0 \neq v \in \mathcal{N}_{-\alpha}$, $R_v : \mathcal{N}_\alpha \to \mathcal{N}_0$ is an injevtive $\mathcal{N}_0$-linear map by (3.37) and (3.74) and (3.86) with $v_2 = e$. So dim $(\mathcal{N}_\alpha/\mathcal{N}_0) = 1$. In particular, dim $(\mathcal{N}_2/\mathcal{N}_0) = 1$. We choose

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$0 \neq u \in \mathcal{N}_1$. Then $R_u : \mathcal{N}_1 \to \mathcal{N}_2$ is an injective $\mathcal{N}_0$-linear map. Hence dim $(\mathcal{N}_1/\mathcal{N}_0) = 1$. Therefore, we get

$$\dim (\mathcal{N}_\alpha/\mathcal{N}_0) = 1 \qquad \text{for } \alpha \in \Delta. \tag{3.91}$$

Since we assume $\hat{\mathcal{N}} \neq \mathcal{N}$, (3.91) implies

$$\mathcal{N}'_\alpha = \mathcal{N}_\alpha \qquad \text{for } \alpha \in \Delta. \tag{3.92}$$

Again we let $\mathbb{F}_1 = \mathcal{N}_0$. We choose $\{0 \neq e_\alpha \in \mathcal{N}_\alpha \mid -1 \neq \alpha \in \Delta\}$ and

$$e_{-1} = e_{-2} \circ e_1. \tag{3.93}$$

We define an algebraic operation "$\cdot$" on $\mathcal{N}$ over $\mathbb{F}_1$

$$e_\alpha \cdot e_\beta = (\beta + 1)^{-1} e_\alpha \cdot (e \circ e_\beta) = (\beta + 1)^{-1} e_\alpha \circ e_\beta \tag{3.94}$$

and

$$e_\alpha \cdot e_{-1} = 2(e_\alpha \cdot e_{-2}) \cdot e_1 \tag{3.95}$$

for $\alpha, \beta \in \Delta$, $\beta \neq -1$. This definition conincides with (3.46) on $\hat{\mathcal{N}} = \sum_{-1 \neq \alpha \in \Delta} \mathcal{N}_\alpha$. For $-1 \neq \alpha \in \Delta$,

$$
\begin{aligned}
e_\alpha \cdot e_{-1} &= 2(e_{-2} \cdot e_\alpha) \cdot e_1 \\
&= (\alpha + 1)^{-1}(e_{-2} \circ e_\alpha) \circ e_1 \\
&= (\alpha + 1)^{-1}(e_{-2} \circ e_1) \circ e_\alpha \\
&= e_{-1} \cdot e_\alpha
\end{aligned}
\tag{3.96}
$$

by (2.31), (3.94), (3.95) and the commutativity of $(\hat{\mathcal{N}}, \cdot)$. Hence $(\mathcal{N}, \cdot)$ is commutative. Thus for $\alpha, \beta, \gamma \in \Delta$ such that $\alpha, \gamma \neq -1$, we have

$$
\begin{aligned}
e_\alpha \cdot (e_\beta \cdot e_\gamma) &= (e_\beta \cdot e_\gamma) \cdot e_\alpha \\
&= (\alpha + 1)^{-1}(\gamma + 1)^{-1}(e_\beta \circ e_\gamma) \circ e_\alpha \\
&= (\alpha + 1)^{-1}(\gamma + 1)^{-1}(e_\beta \circ e_\alpha) \circ e_\gamma \\
&= (e_\beta \cdot e_\alpha) \cdot e_\gamma \\
&= (e_\alpha \cdot e_\beta) \cdot e_\gamma
\end{aligned}
\tag{3.97}
$$

by (2.31). For $\alpha, \beta \in \Delta$ and $\alpha \neq -1$, we ge

$$
\begin{aligned}
(e_\alpha \cdot e_\beta) \cdot e_{-1} &= 2((e_\beta \cdot e_\alpha) \cdot e_{-2}) \cdot e_1 \\
&= -(\alpha + 1)^{-1}((e_\beta \circ e_\alpha) \circ e_{-2}) \circ e_1 \\
&= -(\alpha + 1)^{-1}((e_\beta \circ e_{-2}) \circ e_1) \circ e_\alpha \\
&= 2((e_\beta \cdot e_{-2}) \cdot e_1) \cdot e_\alpha \\
&= (e_\beta \cdot e_{-1}) \cdot e_\alpha \\
&= e_\alpha \cdot (e_\beta \cdot e_{-1}),
\end{aligned}
\tag{3.98}
$$

$$(e_{-1} \cdot e_\beta) \cdot e_{-1} = e_{-1} \cdot (e_{-1} \cdot e_\beta) = e_{-1} \cdot (e_\beta \cdot e_{-1}). \tag{3.99}$$

Thus $(\mathcal{N}, \cdot)$ forms a commutative and associative algebra. According to Theorem 2.1 in [SXZ], we get

$$(\mathcal{N}, \circ) \cong (\mathcal{A}(\Delta, f, \{0\}), \circ_1) \tag{3.100}$$

by (3.62), (3.95) and the fact

$$(\partial + 1)(\mathcal{N}_{-1}) = (-1 + 1)\mathcal{N}_{-1} = \{0\}. \tag{3.101}$$

Case 3. $b \neq 0$ and (3.54) does not hold and $\hat{\mathcal{N}} \neq \mathcal{N}$.

Again we can assume $b = 1$ and $-1 \in \Delta$. Otherwise (3.54) holds. Let

$$\Delta' = \Delta \setminus \mathbb{Z}. \tag{3.102}$$

By (2.32), (3.17), (3.19) and (3.20), the subspace

$$U = \sum_{\alpha \in \Delta'} \mathcal{N}'_\alpha + \sum_{\alpha, \beta \in \Delta'} \mathcal{N}'_\alpha \circ \mathcal{N}'_\beta \tag{3.103}$$

is ideal of $\mathcal{N}$. If $\Delta' \neq \emptyset$, then $U = \mathcal{N}$. So

$$\mathcal{N}'_{-1} \subset \sum_{\alpha, \beta \in \Delta'} \mathcal{N}'_\alpha \circ \mathcal{N}'_\beta, \tag{3.104}$$

that is, (3.54) holds. Hence

$$\Delta \subset \mathbb{Z}. \tag{3.105}$$

The assumption in (3.40) is crucial to the following proof of $1 \in \Delta$. Expression (3.40) implies the equation in (3.18) holds for any $\alpha, \beta \in \Delta$. Since $\sum_{0 \neq m \in \Delta} \mathcal{N}'_m + \sum_{0 \neq m,n \in \Delta} \mathcal{N}'_m \circ \mathcal{N}'_n$ is an ideal by (3.17) and (3.18), we have

$$\mathcal{N} = \sum_{0 \neq m \in \Delta} \mathcal{N}'_m + \sum_{0 \neq m,n \in \Delta} \mathcal{N}'_m \circ \mathcal{N}'_n. \tag{3.106}$$

Hence

$$\mathcal{N}'_0 \subset \sum_{0 \neq m \in \Delta} \mathcal{N}'_m \circ \mathcal{N}'_{-m}. \tag{3.107}$$

Moreover, $\mathcal{N} \circ \mathcal{N} = \mathcal{N}$ by the simplicity of $\mathcal{N}$. Thus

$$\begin{aligned}
\mathcal{N}'_{-1} & \\
\subset & \sum_{m \in \Delta} \mathcal{N}'_m \circ \mathcal{N}'_{-m-1} = \sum_{0,-1 \neq m \in \Delta} \mathcal{N}'_m \circ \mathcal{N}'_{-m-1} + \mathcal{N}'_0 \circ \mathcal{N}'_{-1} + \mathcal{N}'_{-1} \circ \mathcal{N}'_0 \\
\subset & \sum_{0,-1 \neq m \in \Delta} \mathcal{N}'_m \circ \mathcal{N}'_{-m-1} + \sum_{0 \neq m \in \Delta} [(\mathcal{N}'_m \circ \mathcal{N}'_{-m}) \circ \mathcal{N}'_{-1} + \mathcal{N}'_{-1} \circ (\mathcal{N}'_m \circ \mathcal{N}'_{-m})] \\
= & \sum_{0,-1 \neq m \in \Delta} \mathcal{N}'_m \circ \mathcal{N}'_{-m-1} + \sum_{0 \neq m \in \Delta} [(\mathcal{N}'_m \circ \mathcal{N}'_{-1}) \circ \mathcal{N}'_{-m} \\
& + (\mathcal{N}'_{-1} \circ \mathcal{N}'_m) \circ \mathcal{N}'_{-m} + \mathcal{N}'_m \circ (\mathcal{N}'_{-1} \circ \mathcal{N}'_{-m})] \\
= & \sum_{0,-1 \neq m \in \Delta} \mathcal{N}'_m \circ \mathcal{N}'_{-m-1} + (\mathcal{N}'_1 \circ \mathcal{N}'_{-1}) \circ \mathcal{N}'_{-1} \\
& + (\mathcal{N}'_{-1} \circ \mathcal{N}'_1) \circ \mathcal{N}'_{-1} + \mathcal{N}'_1 \circ (\mathcal{N}'_{-1} \circ \mathcal{N}'_{-1})
\end{aligned} \tag{3.108}$$

by (2.31), (2.32), (3.18), (3.40) and (3.107). Since (3.54) fails, we have

$$1 \in \Delta, \tag{3.109}$$

which is important in this case of classification.

Let $0 \neq v \in \mathcal{N}_k$ with $k \in \mathbb{N}$. Set

$$\Psi = \{u \in \mathcal{N} \mid u \circ v = 0\}. \tag{3.110}$$

Then

$$\Psi \circ \mathcal{N} \subset \Psi \tag{3.111}$$

by (2.31). Moreover, for $u \in \Psi$,

$$(e \circ u) \circ v = e \circ (u \circ v) + (u \circ e) \circ v - u \circ (e \circ v) = -ku \circ v = 0 \tag{3.112}$$

by (2.31) and (2.32). So $L_e(\Psi) \subset \Psi$. Thus

$$\Psi = \sum_{m \in \Delta} \Psi_m, \qquad \Psi_m = \Psi \bigcap \mathcal{N}'_m. \tag{3.113}$$

Note that

$$\hat{\mathcal{N}} \circ \Psi = (e \circ \mathcal{N}) \circ \Psi = (e \circ \Psi) \circ \mathcal{N} \subset \Psi \tag{3.114}$$

by (2.31), (3.111) and (3.112). Let $u \in \mathcal{N}'_{-1}$ and $w \in \Psi_m$. We have

$$
\begin{aligned}
&(m+1)(u \circ w) \circ v \\
=\ &(m+1)[u \circ (w \circ v) + (w \circ u) \circ v - w \circ (u \circ v)] \\
=\ &-(m+1)w \circ (u \circ v) \\
=\ &(L_e - m - 1)(w) \circ (u \circ v) - (e \circ w) \circ (u \circ v) \\
=\ &(L_e - m - 1)(w) \circ (u \circ v) - (e \circ (u \circ v)) \circ w \\
=\ &(L_e - m - 1)(w) \circ (u \circ v) - [(e \circ u) \circ v + u \circ (e \circ v) - (u \circ e) \circ v] \circ w \\
=\ &(L_e - m - 1)(w) \circ (u \circ v) - [L_e(u) \circ v + (k+1)u \circ v - u \circ v] \circ w \\
=\ &(L_e - m - 1)(w) \circ (u \circ v) - (L_e(u) \circ v) \circ w - k(u \circ v) \circ w \\
=\ &((L_e - m - 1)(w) \circ u) \circ v + u \circ ((L_e - m - 1)(w) \circ v) - (u \circ (L_e - m - 1)(w)) \circ v \\
&- (L_e(u) \circ w) \circ v - k(u \circ w) \circ v \\
=\ &-[L_e(u) \circ w + u \circ (L_e - m - 1)(w)] \circ v - k(u \circ w) \circ v \tag{3.115}
\end{aligned}
$$

by (2.31), (2.32), (3.40), (3.111) and (3.112). Thus

$$(m+k+1)(u \circ w) \circ v = -[L_e(u) \circ w + u \circ (L_e - m - 1)(w)] \circ v. \tag{3.116}$$

Let n by a positive integer such that $(L_e)^n(u) = (L_e - m - 1)^n(w) = 0$. Then

$$(m+k+1)^{2n}(u \circ w) \circ v = \sum_{j=0}^{2n} \binom{2n}{j}[(L_e)^j(u) \circ (L_e - m - 1)^{2n-j}(w)] \circ v = 0. \tag{3.117}$$

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When $m \neq -k - 1$, we get $(u \circ w) \circ v = 0$. Hence

$$\mathcal{N}'_{-1} \circ \Psi_m \subset \Psi \qquad \text{for} \quad -k - 1 \neq m \in \Delta. \tag{3.118}$$

Set

$$V = \Psi + \sum_{n=1}^{\infty} (L_{\mathcal{N}'_{-1}})^n (\Psi_{-k-1}). \tag{3.119}$$

Note

$$\mathcal{N} \circ \Psi \subset \Psi + L_{\mathcal{N}'_{-1}}(\Psi_{-k-1}) \subset V \tag{3.120}$$

by (3.114) and (3.118). Assume that

$$\mathcal{N} \circ (L_{\mathcal{N}'_{-1}})^n (\Psi_{-k-1}) \subset V. \tag{3.121}$$

$$
\begin{aligned}
\mathcal{N} \circ (L_{\mathcal{N}'_{-1}})^{n+1} (\Psi_{-k-1}) &= (\mathcal{N} \circ \mathcal{N}'_{-1} - \mathcal{N}'_{-1} \circ \mathcal{N}) \circ (L_{\mathcal{N}'_{-1}})^n (\Psi_{-k-1}) \\
&\quad + L_{\mathcal{N}'_{-1}}(\mathcal{N} \circ (L_{\mathcal{N}'_{-1}})^n (\Psi_{-k-1})) \subset V
\end{aligned}
\tag{3.122}
$$

by (2.32) and (3.121). By induction on n, (3.121) holds for any $n \in \mathbb{N}$. Hence V is a left ideal. Furthermore, V is a right ideal by (2.31), (3.111) and (3.121). If $V = \mathcal{N}$, then

$$e \in \Psi \tag{3.123}$$

by (3.18) and the fact $k \geq 0$. Thus

$$0 = e \circ v = (k + 1)v, \tag{3.124}$$

which is absurd. Thus $V = \{0\}$, and we have

$$R_v \text{ is injective} \qquad \text{for } 0 \neq v \in \mathcal{N}_k, \ k \in \mathbb{N}. \tag{3.125}$$

Next we choose $0 \neq u \in \mathcal{N}_n$ for any $-1 \neq n \in \Delta$. Note that $\mathcal{N} \circ u$ is a right ideal. Moreover, for $v_1, v_2 \in \mathcal{N}$, we get

$$
\begin{aligned}
&(e \circ v_1) \circ (v_2 \circ u) \\
={}& (e \circ (v_2 \circ u)) \circ v_1 \\
={}& [(e \circ v_2) \circ u + v_2 \circ (e \circ u) - (v_2 \circ e) \circ u] \circ v_1 \\
={}& [(e \circ v_2) \circ u + (n + 1)v_2 \circ u - v_2 \circ u] \circ v_1 \\
={}& [(e \circ v_2) \circ v_1 + nv_2 \circ v_1] \circ u \subset \mathcal{N} \circ u
\end{aligned}
\tag{3.126}
$$

by (2.31) and (2.32). Hence

$$\mathcal{N}'_m \circ (\mathcal{N} \circ u) \subset \mathcal{N} \circ u \qquad \text{for} \quad -1 \neq m \in \Delta \tag{3.127}$$

by (3.26). Furthermore, for $v_1 \in \mathcal{N}'_{-1}$ and $v_2 \in \mathcal{N}'_m$ with $-1 \neq m \in \Delta$, we have

$$v_1 \circ (v_2 \circ u) = (v_1 \circ v_2) \circ u + v_2 \circ (v_1 \circ u) - (v_2 \circ v_1) \circ u \in \mathcal{N} \circ u \tag{3.128}$$

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by (2.32) and (3.127). Thus

$$\mathcal{N} \circ u + \sum_{j=1}^{\infty} (L_{\mathcal{N}'_{-1}})^j(u) \tag{3.129}$$

is a nonzero ideal of $\mathcal{N}$ by the same arguments as (3.120)-(3.122). So

$$\mathcal{N} = \mathcal{N} \circ u + \sum_{j=1}^{\infty} (L_{\mathcal{N}'_{-1}})^j(u). \tag{3.130}$$

Expressions (3.18) and (3.130) show

$$\mathcal{N}'_m = \mathcal{N}'_{m-n} \circ u \qquad \text{for} \quad n - 1 \le m \in \Delta. \tag{3.131}$$

If $-2 \in \Delta$, we take $n = -2$ in the above and get

$$\mathcal{N}'_1 \circ u = \mathcal{N}'_{-1}, \tag{3.132}$$

which implies (3.54). Thus $\mathcal{N}'_{-2} = \{0\}$. Furthermore, (3.125) with $k = 1$ implies

$$\Delta = \{-1\} \bigcup \mathbb{N}. \tag{3.133}$$

For $u \in \mathcal{N}'_{-1}$ and $v_1, v_2 \in \mathcal{N}_0$, we have

$$
\begin{aligned}
& (u \circ v_1) \circ v_2 - u \circ (v_1 \circ v_2) \\
={} & (v_1 \circ u) \circ v_2 - v_1 \circ (u \circ v_2) \\
={} & ((e \circ v_1) \circ u) \circ v_2 - (e \circ v_1) \circ (u \circ v_2) \\
={} & ((e \circ u) \circ v_1) \circ v_2 - (e \circ (u \circ v_2)) \circ v_1 \\
={} & ((e \circ u) \circ v_1) \circ v_2 - ((e \circ u) \circ v_2) \circ v_1 = 0
\end{aligned}
\tag{3.134}
$$

by (2.31), (2.32), (3.77) and (3.83). Moreover, for $u_1 \in \mathcal{N}'_{-1}$, $u_2 \in \mathcal{N}_\beta$ with $\beta \ne -1$ and $v_1, v_2 \in \mathcal{N}_0$, we get

$$
\begin{aligned}
& (u_1 \circ v_1) \circ (u_2 \circ v_2) \\
={} & (u_1 \circ (u_2 \circ v_2)) \circ v_1 \\
={} & [(u_1 \circ u_2) \circ v_2 + u_2 \circ (u_1 \circ v_2) - (u_2 \circ u_1) \circ v_2] \circ v_1 \\
={} & [(u_1 \circ u_2) \circ v_2 + (u_2 \circ u_1) \circ v_2 - (u_2 \circ u_1) \circ v_2] \circ v_1 \\
={} & ((u_1 \circ u_2) \circ v_2) \circ v_1 \\
={} & (u_1 \circ u_2) \circ (v_2 \circ v_1) \\
={} & (u_1 \circ u_2) \circ (v_1 \circ v_2)
\end{aligned}
\tag{3.135}
$$

by (2.31), (2.32), (3.40), (3.79), (3.85), (3.134) and the commutativity of $(\mathcal{N}_0, \cdot)$. Futhermore,

$$\mathcal{N}'_{-1} \circ \mathcal{N}'_{-1} = \{0\} \tag{3.136}$$

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by (3.18) and (3.133). Thus (3.86) holds in this case by (3.80), (3.85), (3.135) and (3.136). Therefore, $(\mathcal{N}_0, \circ)$ forms a field and (3.92) holds. Pick any $0 \neq e' \in \mathcal{N}_{-1}$, then

$$e' \circ e' = 0 \tag{3.137}$$

by (3.136) and

$$e' \circ \mathcal{N} = \mathcal{N} \tag{3.138}$$

by (3.86), (3.92), (3.125) and (3.138). By (2.31), (3.137) and (3.138), $R_{e'} = 0$. Replacing e by e', we go back to Case 1.

This completes the proof of Theorem 3.4.   □

4   Classification of Irreducible Modules

In this section, we shall classify all the irreducible modules of a certain infinite-dimensional simple Novikov algebras with an idempotent element whose left action is locally finite.

As usual, a submodule of a module M of a Novikov algebra $(\mathcal{N}, \circ)$ is a subspace V of M such that

$$u \circ V, \ V \circ u \subset V \qquad \text{for} \ u \in \mathcal{N}. \tag{4.1}$$

The module M is called *irreducible* if it does not contain any proper nonzero submodule. First we present a Lemma due to Osborn [O4] and gave the proof for the reader's convenience. Recall the notations in (3.6).

Lemma 4.1 (Osbron). *Let M be an irreducible module of a Novikov algebra $(\mathcal{N}, \circ)$. Suppose that $R_M([\mathcal{N}, \mathcal{N}]^-) \neq \{0\}$. If T is a polynomial of right multiplication operators on $\mathcal{N}$ such that $T(\mathcal{N}) = \{0\}$, then the operator T' obtained from T with R_u replaced by $R_M(u)$ also satisfy $T'(M) = \{0\}$.*

Proof. Note that $\mathcal{N} \circ M$ is a submodule of M by (2.31). If $M = \mathcal{N} \circ M$, then

$$T'(M) = T'(\mathcal{N} \circ M) = T(\mathcal{N}) \circ M = \{0\} \tag{4.2}$$

by (2.31). So the lemma holds.

If $M \neq \mathcal{N} \circ M$, then $\mathcal{N} \circ M = \{0\}$ by the irreducibility of M. For $u_1, u_2 \in \mathcal{N}$ and $w \in M$, we have:

$$(w \circ u_1) \circ u_2 - w \circ (u_1 \circ u_2) = (u_1 \circ w) \circ u_2 - u_1 \circ (w \circ u_2) = 0 \tag{4.3}$$

by (2.32). Moreover,

$$\begin{aligned} R_M([u_1, u_2]^-)(w) &= w \circ (u_1 \circ u_2 - u_2 \circ u_1) \\ &= (w \circ u_1) \circ u_2 - (w \circ u_2) \circ u_1 \\ &= (w \circ u_2) \circ u_1 - (w \circ u_2) \circ u_1 = 0 \end{aligned} \tag{4.4}$$

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by (2.31) and (4.3). Since w is arbitrary, $R_M([u_1, u_2]^-) = 0$. Hence $R_M([\mathcal{N}, \mathcal{N}]^-) = \{0\}$, which contradicts our assumption. $\qquad \square$

Next we shall give the constructions of irreducible modules. Take J to be the additive semi-group $\{0\}$ or $\mathbb{N}$. Let $\mathcal{A}$ be a vector space with a basis

$$\{u_{\alpha,i} \mid \alpha \in \mathbb{F}, i \in J\}. \tag{4.5}$$

Define the operation "$\cdot$" on $\mathcal{A}$ by

$$u_{\alpha,i} \cdot u_{\beta,j} = u_{\alpha+\beta,i+j} \qquad \text{for } \alpha, \beta \in \mathbb{F}, \ i, j \in J. \tag{4.6}$$

Then $(\mathcal{A}, \cdot)$ forms a commutative associative algebra with the identity element $1 = u_{0,0}$. We define the map $\partial : \mathcal{A} \to \mathcal{A}$ by:

$$\partial(u_{\alpha,j}) = \alpha u_{\alpha,j} + j u_{\alpha,j-1} \qquad \text{for } \alpha \in \mathbb{F}, \ j \in J. \tag{4.7}$$

Let Δ be an additive subgroup of $\mathbb{F}$ such that $J + \Delta \neq \{0\}$. Set

$$\mathcal{N} = \sum_{\alpha \in \Delta, i \in J} \mathbb{F} u_{\alpha,i}. \tag{4.8}$$

Next, for any fixed element $\xi \in \mathcal{N}$, we define the operation "$\circ$" on $\mathcal{A}$ by

$$u \circ v = u \cdot \partial(v) + \xi \cdot u \cdot v \qquad \text{for } u, v \in \mathcal{A}. \tag{4.9}$$

By Theorem 2.9 in [X2], $(\mathcal{A}, \circ)$ forms a simple Novikov algebra and $(\mathcal{N}, \circ)$ forms a simple subalgebra of $(\mathcal{A}, \circ)$. For $\lambda \in \mathbb{F}$, we set

$$M(\lambda) = \sum_{\alpha \in \Delta, i \in J} \mathbb{F} u_{\alpha+\lambda,i} \tag{4.10}$$

Expression (4.9) shows

$$\mathcal{N} \circ M(\lambda), \ M(\lambda) \circ \mathcal{N} \subset M(\lambda). \tag{4.11}$$

Thus $M(\lambda)$ forms an $\mathcal{N}$-module. In fact, by a similar proof as that of Theorem 2.9 in [X2], we obtain:

Theorem 4.2. *The $\mathcal{N}$-module $M(\lambda)$ is irreducible.*

A natural question is to what extent the modules $\{M(\lambda) \mid \lambda \in \mathbb{F}\}$ cover the irreducible modules of $\mathcal{N}$. Up to this point, we are not be able to answer this for a general element $\xi \in \mathcal{N}$. The following is our second main theorem in this paper.

Theorem 4.3. *If $\xi = b \in \mathbb{F}$, then any irreducible $\mathcal{N}$-module M with locally finite $L_M(u_{0,0})$ is isomorphic to $M(\lambda)$ for some $\lambda \in \mathbb{F}$.*

Proof. Assume $\xi = b$. Then (4.9) becomes

$$u_{\alpha,i} \circ u_{\beta,j} = (\beta + b)u_{\alpha+\beta,i+j} + ju_{\alpha+\beta,i+j-1} \qquad \text{for } \alpha, \beta \in \mathbb{F},\ i, j \in J. \tag{4.12}$$

In particular,

$$u_{0,0} \circ u_{\gamma,l} = (\gamma + b)u_{\gamma,l} + lu_{\gamma,l-1} \qquad \text{for } \gamma \in \Delta,\ l \in J \tag{4.13}$$

and

$$u_{\alpha,i} \circ u_{\lambda,0} = (\lambda + b)u_{\alpha+\lambda,i} \qquad \text{for } \alpha, \lambda \in \mathbb{F},\ i \in J. \tag{4.14}$$

Expression (4.14) implies

$$R_{u_{0,0}} = b\mathrm{Id}_{\mathcal{A}}. \tag{4.15}$$

Let M be an $\mathcal{N}$-*module* M with locally finite $L_M(u_{0,0})$. The local finiteness of $L_M(u_{0,0})$ implies the existence of its eigenvectors in M.

Case 1. The operator $L_M(u_{0,0})$ has an eigenvector with a nonzero eigenvalue.

Let w be the eigenvector and let $\lambda + b$ be the coresponding nonzero eigenvalue. Set

$$M' = \mathcal{N} \circ w = \sum_{\alpha \in \Delta, i \in J} \mathbb{F}u_{\alpha,i} \circ w. \tag{4.16}$$

In particular, $w \in M'$. For $\alpha \in \Delta$, $i \in J$ and $u \in \mathcal{N}$,

$$(u_{\alpha,i} \circ w) \circ u = (u_{\alpha,i} \circ u) \circ w \in M' \tag{4.17}$$

by (2.31). Note that

$$\begin{aligned}
u_{0,0} \circ (u_{\alpha,i} \circ w) &= (u_{0,0} \circ u_{\alpha,i}) \circ w + u_{\alpha,i} \circ (u_{0,0} \circ w) - (u_{\alpha,i} \circ u_{0,0}) \circ w \\
&= (\alpha + b)u_{\alpha,i} \circ w + iu_{\alpha,i-1} \circ w + (\lambda + b)u_{\alpha,i} \circ w - bu_{\alpha,i} \circ w \\
&= (\alpha + \lambda + b)u_{\alpha,i} \circ w + iu_{\alpha,i-1} \circ w \tag{4.18}
\end{aligned}$$

by (2.32). Moreover,

$$\begin{aligned}
& (u_{0,0} \circ u) \circ (u_{\alpha,i} \circ w) \\
=\ & (u_{0,0} \circ (u_{\alpha,i} \circ w)) \circ u \\
=\ & ((\alpha + \lambda + b)u_{\alpha,i} \circ w + iu_{\alpha,i-1} \circ w) \circ u \\
=\ & (\alpha + \lambda + b)(u_{\alpha,i} \circ u) \circ w + i(u_{\alpha,i-1} \circ u) \circ w \tag{4.19}
\end{aligned}$$

by (2.31) and (4.18). Futhermore, (4.13) shows

$$u_{0,0} \circ \mathcal{N} = \mathcal{N} \qquad \text{if } J = \mathbb{N} \text{ or } b \notin \Delta. \tag{4.20}$$

Thus M' is a nonzero submodule of M if $J = \mathbb{N}$ or $b \notin \Delta$ by (4.17) and (4.19). Assume that $J = \{0\}$ and $b \in \Delta$. Since we assume $J + \Delta \neq \{0\}$, we have $\Delta \neq \{0\}$. In this case,

$$u_{\alpha,0} \in u_{0,0} \circ \mathcal{N} \qquad \text{for } -b \neq \alpha \in \Delta \tag{4.21}$$

by (4.13). Since char $\mathbb{F} = 0$, we have $|\Delta| = \infty$. There exists $\gamma \in \Delta$ such that $\gamma \neq 0, -b$. Since $u_{\gamma-b,0}, u_{\gamma,0} \in u_{0,0} \circ \mathcal{N}$ by (4.21). Hence

$$
\begin{aligned}
u_{-b,0} \circ (u_{\alpha,0} \circ w) &= (\gamma + b)^{-1}(u_{-\gamma-b,0} \circ u_{\gamma,0}) \circ (u_{\alpha,0} \circ w) \\
&= (\gamma + b)^{-1}(u_{-\gamma-b,0} \circ (u_{\gamma,0} \circ w)) \circ u_{\gamma,0} \\
&\in (\mathcal{N} \circ w) \circ u_{\gamma,0} \\
&= (\mathcal{N} \circ u_{\gamma,0}) \circ w \\
&\subset \mathcal{N} \circ w \\
&= M'
\end{aligned}
\tag{4.22}
$$

for $\alpha \in \Delta$ by (2.31), (4.14) and (4.19). Therefore, M' is again a nonzero submodule of M. The irreducibility of M shows

$$
M = M'.
\tag{4.23}
$$

We define a linear map $\sigma : M(\lambda) \to M$ by

$$
\sigma(u_{\alpha+\lambda,i}) = u_{\alpha,i} \circ w \qquad \text{for } \alpha \in \Delta,\ i \in J.
\tag{4.24}
$$

By (4.12) and (4.17),

$$
\begin{aligned}
\sigma(u_{\alpha+\lambda,i} \circ u_{\beta,j}) &= \sigma((\beta + b)u_{\alpha+\beta+\lambda,i+j} + j u_{\alpha+\beta+\lambda,i+j-1}) \\
&= (\beta + b)\sigma(u_{\alpha+\beta+\lambda,i+j}) + j\sigma(u_{\alpha+\beta+\lambda,i+j-1}) \\
&= (\beta + b)u_{\alpha+\beta,i+j} \circ w + j u_{\alpha+\beta,i+j-1} \circ w \\
&= [(\beta + b)u_{\alpha+\beta,i+j} + j u_{\alpha+\beta,i+j-1}] \circ w \\
&= (u_{\alpha,i} \circ u_{\beta,j}) \circ w \\
&= (u_{\alpha,i} \circ w) \circ u_{\beta,j} \\
&= \sigma(u_{\alpha+\lambda,i}) \circ u_{\beta,j}
\end{aligned}
\tag{4.25}
$$

for $\alpha, \beta \in \Delta$ and $i, j \in J$. Moreover, (4.13) and (4.17) show

$$
\begin{aligned}
\sigma(u_{0,0} \circ u_{\alpha+\lambda,i}) &= \sigma((\alpha + \lambda + b)u_{\alpha+\lambda,i} + i u_{\alpha+\lambda,i-1}) \\
&= (\alpha + \lambda + b)\sigma(u_{\alpha+\lambda,i}) + i\sigma(u_{\alpha+\lambda,i-1}) \\
&= (\alpha + \lambda + b)u_{\alpha,i} \circ w + i u_{\alpha,i-1} \circ w \\
&= u_{0,0} \circ (u_{\alpha,i} \circ w) \\
&= u_{0,0} \circ \sigma(u_{\alpha+\lambda,i})
\end{aligned}
\tag{4.26}
$$

for $\alpha \in \Delta$ and $i \in J$. Furthermore,

$$
\begin{aligned}
\sigma((u_{0,0} \circ u_{\beta,j}) \circ u_{\alpha+\lambda,i}) &= \sigma((u_{0,0} \circ u_{\alpha+\lambda,i}) \circ u_{\beta,j}) \\
&= \sigma(u_{0,0} \circ u_{\alpha+\lambda,i}) \circ u_{\beta,j} \\
&= (u_{0,0} \circ \sigma(u_{\alpha+\lambda,i})) \circ u_{\beta,j} \\
&= (u_{0,0} \circ u_{\beta,j}) \circ \sigma(u_{\alpha+\lambda,i})
\end{aligned}
\tag{4.27}
$$

for $\alpha, \beta \in \Delta$ and $i, j \in J$. Thus σ is an $\mathcal{N}$-module homomorphism when $J = \mathbb{N}$ or $b \notin \Delta$ by (4.20). If $J = \{0\}$ and $b \in \Delta$, we choose $\gamma \in \Delta$ such that $\gamma \neq 0, -b$ and get

$$
\begin{aligned}
\sigma(u_{-b,0} \circ u_{\alpha+\lambda,0}) &= (\gamma+b)^{-1}\sigma((u_{-\gamma-b,0} \circ u_{\gamma,0}) \circ u_{\alpha+\lambda,0}) \\
&= (\gamma+b)^{-1}\sigma((u_{-\gamma-b,0} \circ u_{\alpha+\lambda,0}) \circ u_{\gamma,0}) \\
&= (\gamma+b)^{-1}\sigma(u_{-\gamma-b,0} \circ u_{\alpha+\lambda,0}) \circ u_{\gamma,0} \\
&= (\gamma+b)^{-1}(u_{-\gamma-b,0} \circ \sigma(u_{\alpha+\lambda,0})) \circ u_{\gamma,0} \\
&= (\gamma+b)^{-1}(u_{-\gamma-b,0} \circ u_{\gamma,0}) \circ \sigma(u_{\alpha+\lambda,0}) \\
&= u_{-b,0} \circ \sigma(u_{\alpha+\lambda,0}) \qquad (4.28)
\end{aligned}
$$

by (4.21) and (4.27). Again σ is an $\mathcal{N}$-module homomorphism. Since $M(\lambda)$ is irreducible and $M \neq \{0\}$, σ must be an $\mathcal{N}$-module isomorphism.

Case 2. The operator $L_M(u_{0,0})$ has only zero eigenvalue.

By (4.12),

$$
[u_{\alpha,i}, u_{\beta,j}]^- = (\beta - \alpha)u_{\alpha+\beta,i+j} + (j-i)u_{\alpha+\beta,i+j-1} \qquad \text{for } \alpha, \beta \in \Delta, \ i, j \in J. \qquad (4.29)
$$

In particular,

$$
[u_{0,0}, u_{\beta,j}]^- = \beta u_{\beta,j} + j u_{\beta,j-1} \qquad \text{for } \beta \in \Delta, \ j \in J. \qquad (4.30)
$$

Hence

$$
[u_{0,0}, \mathcal{N}]^- = \mathcal{N} \qquad (4.31)
$$

if $J = \mathbb{N}$. When $J = \{0\}$, $\Delta \neq \{0\}$ by our assumption. Note that (4.30) shows

$$
u_{\alpha,0} \in [u_{0,0}, \mathcal{N}]^- \qquad \text{for } 0 \neq \alpha \in \Delta. \qquad (4.32)
$$

Moreover,

$$
[u_{-\alpha,0}, u_{\alpha,0}]^- = 2\alpha u_{0,0} \qquad \text{for } \alpha \in \Delta. \qquad (4.33)
$$

Thus we always have

$$
[\mathcal{N}, \mathcal{N}]^- = \mathcal{N}. \qquad (4.34)
$$

If $R_M([\mathcal{N}, \mathcal{N}]^-) = \{0\}$, then $R_M(\mathcal{N}) = 0$. So $\mathcal{N} \circ M \neq \{0\}$ because M is not trivial. Since $\mathcal{N} \circ M$ is a submodule of M, $M = \mathcal{N} \circ M$. But

$$
M = \mathcal{N} \circ M = (\mathcal{N} \circ \mathcal{N}) \circ M = (\mathcal{N} \circ M) \circ \mathcal{N} = \{0\}, \qquad (4.35)
$$

which contradicts to the non-triviality of M. Thus $R_M([\mathcal{N}, \mathcal{N}]^-) \neq \{0\}$. By Lemma 4.1 and (4.15), we have:

$$
R_M(u_{0,0}) = b\mathrm{Id}_M. \qquad (4.36)
$$

Again we let w be an eigenvector of $L_{u_{0,0}}$ (remember the corresponding eigenvalue is 0). Then we have

$$
\begin{aligned}
u_{0,0} \circ (w \circ u_{\alpha,i}) &= (u_{0,0} \circ w) \circ u_{\alpha,i} + w \circ (u_{0,0} \circ u_{\alpha,i}) - (w \circ u_{0,0}) \circ u_{\alpha,i} \\
&= w \circ ((\alpha + b)u_{\alpha,i} + iu_{\alpha,i-1}) - bw \circ u_{\alpha,i} \\
&= \alpha w \circ u_{\alpha,i} + iw \circ u_{\alpha,i-1}
\end{aligned}
\tag{4.37}
$$

for $\alpha \in \Delta$ and $i \in J$ by (2.32), (4.13) and (4.36). In particular, $w \circ u_{\alpha,0}$ is an eigenvector with the eigenvalue α if it is not zero. Hence by our assumption of the zero eigenvalue of $L_M(u_{0,0})$, we have

$$
w \circ u_{\alpha,0} = 0 \qquad \text{for } 0 \neq \alpha \in \Delta.
\tag{4.38}
$$

Moreover, (4.37) shows that $w \circ u_{\alpha,1}$ is an eigenvector with the eigenvalue α if it is not zero. So

$$
w \circ u_{\alpha,1} = 0 \qquad \text{for } 0 \neq \alpha \in \Delta.
\tag{4.39}
$$

Continuing this process, we can prove

$$
w \circ u_{\alpha,i} = 0 \qquad \text{for } 0 \neq \alpha \in \Delta, \ i \in J.
\tag{4.40}
$$

Note that

$$
(u_{0,0} \circ u_{\alpha,i}) \circ w = (u_{0,0} \circ w) \circ u_{\alpha,i} = 0 \qquad \text{for } \alpha \in \Delta, \ i \in J.
\tag{4.41}
$$

So

$$
\mathcal{N} \circ w = \{0\} \qquad \text{if } J = \mathbb{N} \text{ or } b \notin \Delta
\tag{4.42}
$$

by (4.20). Assume that $J = \{0\}$ and $b \in \Delta$.

$$
u_{\alpha,0} \circ w = 0 \qquad \text{for } -b \neq \alpha \in \Delta
\tag{4.43}
$$

by (4.21). Moreover, we choose $\gamma \in \Delta$ such that $\gamma \neq 0, -b$ and get

$$
u_{-b,0} \circ w = (\gamma + b)^{-1}(u_{-\gamma-b} \circ u_{\gamma,0}) \circ w = (\gamma + b)^{-1}(u_{-\gamma-b} \circ w) \circ u_{\gamma,0} = 0
\tag{4.44}
$$

by (2.31). Thus we always have

$$
\mathcal{N} \circ w = \{0\}.
\tag{4.45}
$$

If $\Delta \neq \{0\}$, then we choose any $0, -b \neq \beta \in \Delta$. We get

$$
\begin{aligned}
&(\beta + b)w \circ u_{0,j} + jw \circ u_{0,j-1} \\
={} & w \circ (u_{-\beta,0} \circ u_{\beta,j}) \\
={} & (w \circ u_{-\beta,0}) \circ u_{\beta,j} + u_{-\beta,0} \circ (w \circ u_{\beta,j}) - (u_{-\beta,0} \circ w) \circ u_{\beta,j} = 0
\end{aligned}
\tag{4.46}
$$

by (2.32), (4.40) and (4.45). By induction on j, we get

$$
w \circ u_{0,j} = 0 \qquad \text{for } j \in J.
\tag{4.47}
$$

So $\mathbb{F}w$ is a trivial submodule by (4.40), (4.45) and (4.47), which contradicts to the irreduciblity of M. Thus $\Delta = \{0\}$, which implies $J = \mathbb{N}$ by our assumption.

We redenote $u_{0,j}$ by u_j for $j \in \mathbb{N}$. Now (4.37) becomes

$$u_0 \circ (w \circ u_i) = iw \circ u_{i-1} \qquad \text{for } i \in \mathbb{N}. \tag{4.48}$$

Note that for $i, j, l \in \mathbb{N}$ and $\gamma \in \mathbb{F}$, we have

$$
\begin{aligned}
R_{u_j} R_{u_i}(u_{\gamma,l}) &= R_{u_j}(bu_{\gamma,i+l} + iu_{\gamma,i+l-1}) \\
&= b^2 u_{\gamma,i+j+l} + (i+j)bu_{\gamma,i+j+l-1} + iju_{\gamma,i+j+l-2}
\end{aligned} \tag{4.49}
$$

by (4.12). If $b = 0$ and $i + j > 1$, the above expression shows

$$R_{u_j} R_{u_i} = ij(i+j-1)^{-1} R_{u_{i+j-1}} \tag{4.50}$$

on $\mathcal{A}$, and in particular on $\mathcal{N}$. Assume $b \neq 0$. Then

$$R_{u_j} R_{u_i} = bR_{u_{i+j}} + ijb^{-1}\left(R_{u_{i+j-1}} + \sum_{k=1}^{i+j-2} (-1)^k b^{-k}(i+j-1)\cdots(i+j-k)R_{u_{i+j-k-1}}\right) \tag{4.51}$$

on $\mathcal{A}$ and in particular on $\mathcal{N}$. By Lemma 4.1, we must have

$$R_M(u_j)R_M(u_i) = ij(i+j-1)^{-1} R_M(u_{i+j-1}) \qquad \text{for } i, j \in \mathbb{N}, \ i+j > 1 \tag{4.52}$$

when $b = 0$ and

$$
\begin{aligned}
R_M(u_j)R_M(u_i) &= bR_M(u_{i+j}) + ijb^{-1}(R_M(u_{i+j-1}) \\
&\quad + \sum_{k=1}^{i+j-2} (-1)^k b^{-k}(i+j-1)\cdots(i+j-k)R_M(u_{i+j-k-1}))
\end{aligned} \tag{4.53}
$$

for $i, j \in \mathbb{N}$ when $b \neq 0$.

Set

$$M' = \sum_{j=0}^{\infty} \mathbb{F}w \circ u_j. \tag{4.54}$$

Then

$$(w \circ u_0) \circ u_j = bw \circ u_j \qquad \text{for } j \in \mathbb{N}. \tag{4.55}$$

For $0 < i, j \in \mathbb{N}$, we have

$$(w \circ u_i) \circ u_j = ij(i+j-1)^{-1}w \circ u_{i+j-1} \qquad \text{when } b = 0 \tag{4.56}$$

by (4.52) and

$$
\begin{aligned}
(w \circ u_i) \circ u_j &= bw \circ u_{i+j} + ijb^{-1}(w \circ u_{i+j-1} \\
&\quad + \sum_{k=1}^{i+j-2} (-1)^k b^{-k}(i+j-1)\cdots(i+j-k)w \circ u_{i+j-k-1})
\end{aligned} \tag{4.57}
$$

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when $b \neq 0$. So
$$M' \circ \mathcal{N} \subset M' \tag{4.58}$$
by (4.36), (4.55)-(4.57). Furthermore,
$$(u_0 \circ u_j) \circ (w \circ u_i) = (u_0 \circ (w \circ u_i)) \circ u_j = i(w \circ u_{i-1}) \circ u_j \in M' \tag{4.59}$$
for $0 < i, j \in \mathbb{N}$ by (2.31), (4.48) and (4.57). Since $u_0 \circ \mathcal{N} = \mathcal{N}$ by (4.13), M' is a submodule of M by (4.45), (4.48), (4.58) and (4.59). When $b \neq 0$, $w = b^{-1}w \circ u_0 \in M'$. So $M' \neq \{0\}$. If $b = 0$, then $w \circ u_0 = 0$. If $M' = \{0\}$, then $\mathbb{F}w$ forms a trivial submodule by (4.45), which contradicts to the irreducibility of M. Hence, we always have $M' \neq \{0\}$. Thus
$$M' = M. \tag{4.60}$$

Assume $b \neq 0$. Observe that (4.12) implies
$$u_{-b,0} \circ u_i = bu_{-b,i} + iu_{-b,i-1} \qquad \text{for } i \in \mathbb{N}. \tag{4.61}$$
Hence
$$\{u_{-b,0} \circ u_i \mid i \in \mathbb{N}\} \tag{4.62}$$
is a basis of $M(-b)$. We define a linear map $\sigma : M(-b) \to M$ by
$$\sigma(u_{-b,0} \circ u_i) = w \circ u_i \qquad \text{for } i \in \mathbb{N}. \tag{4.63}$$
By (4.13) and (4.61),
$$
\begin{aligned}
u_0 \circ (u_{-b,0} \circ u_i) &= u_0 \circ (bu_{-b,i} + iu_{-b,i-1}) \\
&= i(bu_{-b,i-1} + (i-1)u_{-b,i-2}) \\
&= i(u_{-b,0} \circ u_{i-1}).
\end{aligned}
\tag{4.64}
$$
Moreover,
$$
\begin{aligned}
& \sigma((u_{-b,0} \circ u_i) \circ u_j) \\
=\ & \sigma[(bR_{u_{i+j}} + ijb^{-1}(R_{u_{i+j-1}} + \sum_{k=1}^{i+j-2} (-1)^k b^{-k} \\
& (i+j-1)\cdots(i+j-k)R_{u_{i+j-k-1}}))(u_{-b,0})] \\
=\ & [bR_M(u_{i+j}) + ijb^{-1}(R_M(u_{i+j-1}) + \sum_{k=1}^{i+j-2} (-1)^k b^{-k} \\
& (i+j-1)\cdots(i+j-k)R_M(u_{i+j-k-1}))](w) \\
=\ & (w \circ u_i) \circ u_j \\
=\ & \sigma(u_{-b,0} \circ u_i) \circ u_j
\end{aligned}
\tag{4.65}
$$

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for $i, j \in \mathbb{N}$ by (4.51) and (4.53). Furthermore,

$$
\begin{aligned}
\sigma((u_0 \circ u_j) \circ (u_{-b,0} \circ u_i)) &= \sigma((u_0 \circ (u_{-b,0} \circ u_i)) \circ u_j) \\
&= i\sigma((u_{-b,0} \circ u_{i-1}) \circ u_j) \\
&= i\sigma(u_{-b,0} \circ u_{i-1}) \circ u_j \\
&= i(w \circ u_{i-1}) \circ u_j \\
&= (u_0 \circ (w \circ u_i)) \circ u_j \\
&= (u_0 \circ u_j) \circ (w \circ u_{i-1}) \\
&= (u_0 \circ u_j) \circ \sigma(u_{-b,0} \circ u_i)
\end{aligned}
\tag{4.66}
$$

by (2.31), (4.48), (4.64) and (4.65). Since $u_0 \circ \mathcal{N} = \mathcal{N}$, σ is an $\mathcal{N}$-module homomorphism by the above two expressions. Therefore, σ is an isomorphism by the irreducibility of $M(-b)$.

Finally, we assume $b = 0$. In this case.

$$
u_0 \circ u_j = j u_{j-1} \qquad \text{for } j \in \mathbb{N}.
\tag{4.67}
$$

Hence

$$
\{u_0 \circ u_j \mid j \in \mathbb{N}\}
\tag{4.68}
$$

forms a basis of $\mathcal{N} = M(0)$. We define a linear map $\sigma : \mathcal{N} \to M$ by

$$
\sigma(u_0 \circ u_i) = w \circ u_i \qquad \text{for } i \in \mathbb{N}.
\tag{4.69}
$$

By (4.67),

$$
u_0 \circ (u_0 \circ u_i) = i u_0 \circ u_{i-1}.
\tag{4.70}
$$

Moreover, for $0 < i, j \in \mathbb{N}$,

$$
\begin{aligned}
\sigma((u_0 \circ u_i) \circ u_j) &= ij(i+j-1)^{-1}\sigma(u_0 \circ u_{i+j-1}) \\
&= ij(i+j-1)^{-1}w \circ u_{i+j-1} \\
&= (w \circ u_i) \circ u_j \\
&= \sigma(w \circ u_i) \circ u_j
\end{aligned}
\tag{4.71}
$$

by (4.52). In addition, (4.15) and (4.36) show

$$
\sigma((u_0 \circ u_i) \circ u_0) = \sigma(0) = 0 = \sigma(u_0 \circ u_i) \circ u_0.
\tag{4.72}
$$

Furthermore,

$$
\begin{aligned}
\sigma((u_0 \circ u_j) \circ (u_0 \circ u_i)) &= \sigma((u_0 \circ (u_0 \circ u_i)) \circ u_j) \\
&= i\sigma((u_0 \circ u_{i-1}) \circ u_j) \\
&= i\sigma(u_0 \circ u_{i-1}) \circ u_j
\end{aligned}
$$

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$$
\begin{aligned}
&= i(w \circ u_{i-1}) \circ u_j \\
&= (u_0 \circ (w \circ u_i)) \circ u_j \\
&= (u_0 \circ u_j) \circ (w \circ u_{i-1}) \\
&= (u_0 \circ u_j) \circ \sigma(u_{-b,0} \circ u_i) \qquad\qquad (4.73)
\end{aligned}
$$

for $i, j \in \mathbb{N}$ by (2.31), (4.48) and (4.70)-(4.72). Since $u_0 \circ \mathcal{N} = \mathcal{N}$, σ is an $\mathcal{N}$-module homomorphism by the above three expressions, which must be an isomorphism because $M(0) \cong \mathcal{N}$ is an irreducible $\mathcal{N}$-module. $\qquad \square$

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